Theory of Structures

III year-I Semester

Unit-I

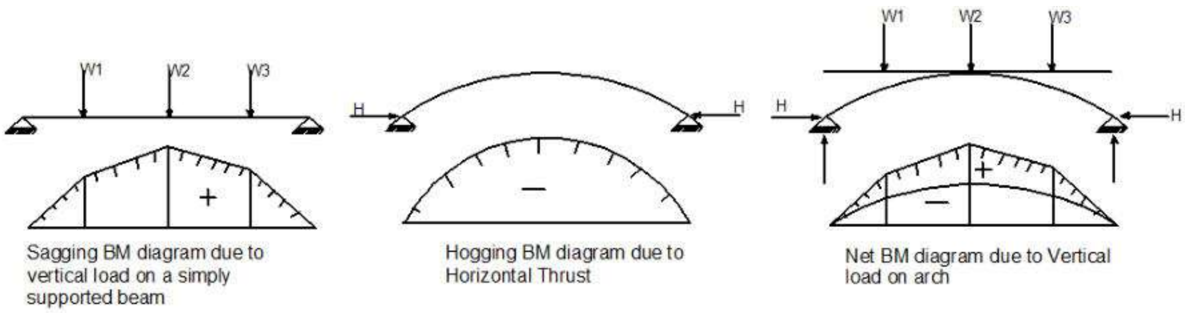
Arches

Learning Material

Arch:

An arch may be visualized as a curved beam in elevation with convexity upward which is restrained at it ends from spreading outwards under the action of download vertical loads.

The inward horizontal reactions induced by the end restraints produce hogging moments in the arch whichwill counter act the static sagging moments set up by the vertical loads.



The consequent reduction in the net moments which is responsible for a significantly higher load carrying capacity of an arch as compared to the corresponding beam , presents the arch as an effective option in construction of bridges, buildings etc.,

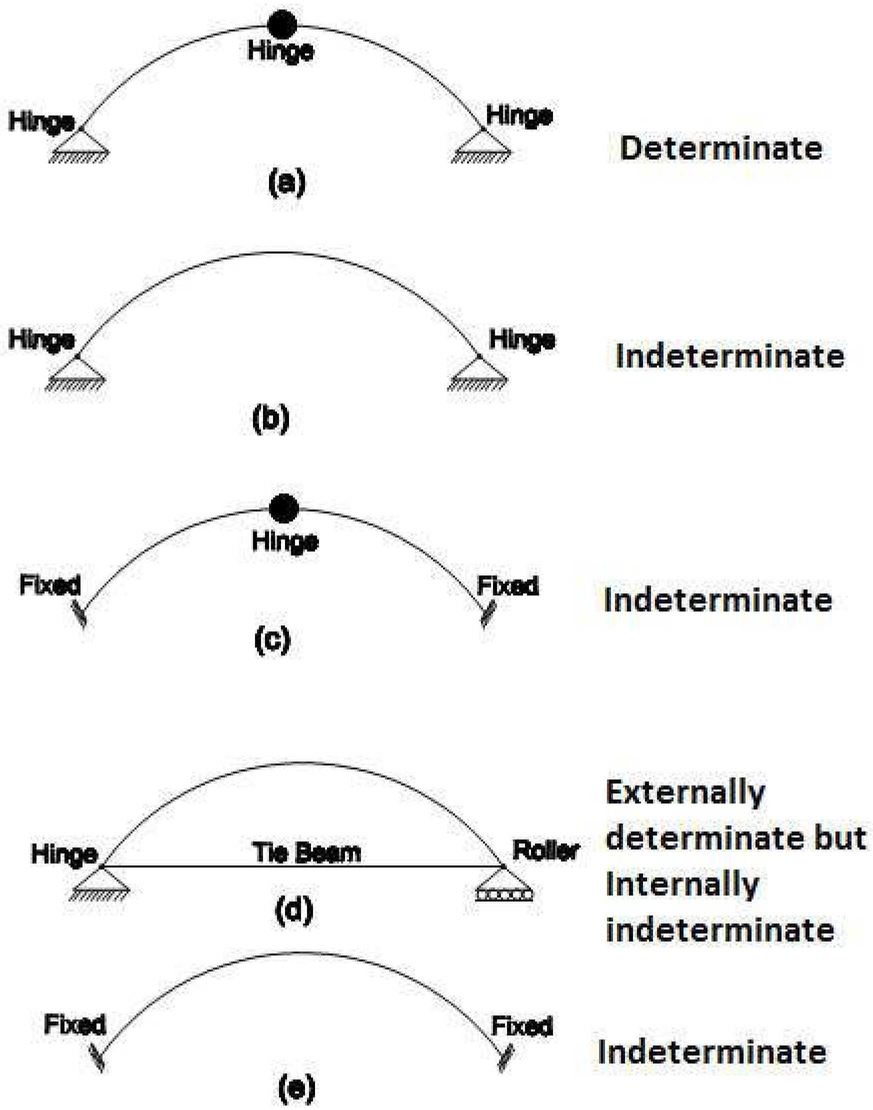
Since the transverse loading at any section normal to the axis of the girder is at an angle to the normal face, an arch is subjected to three restraining forces

1. Normal thrust
2. Radial shear and
3. Bending moment

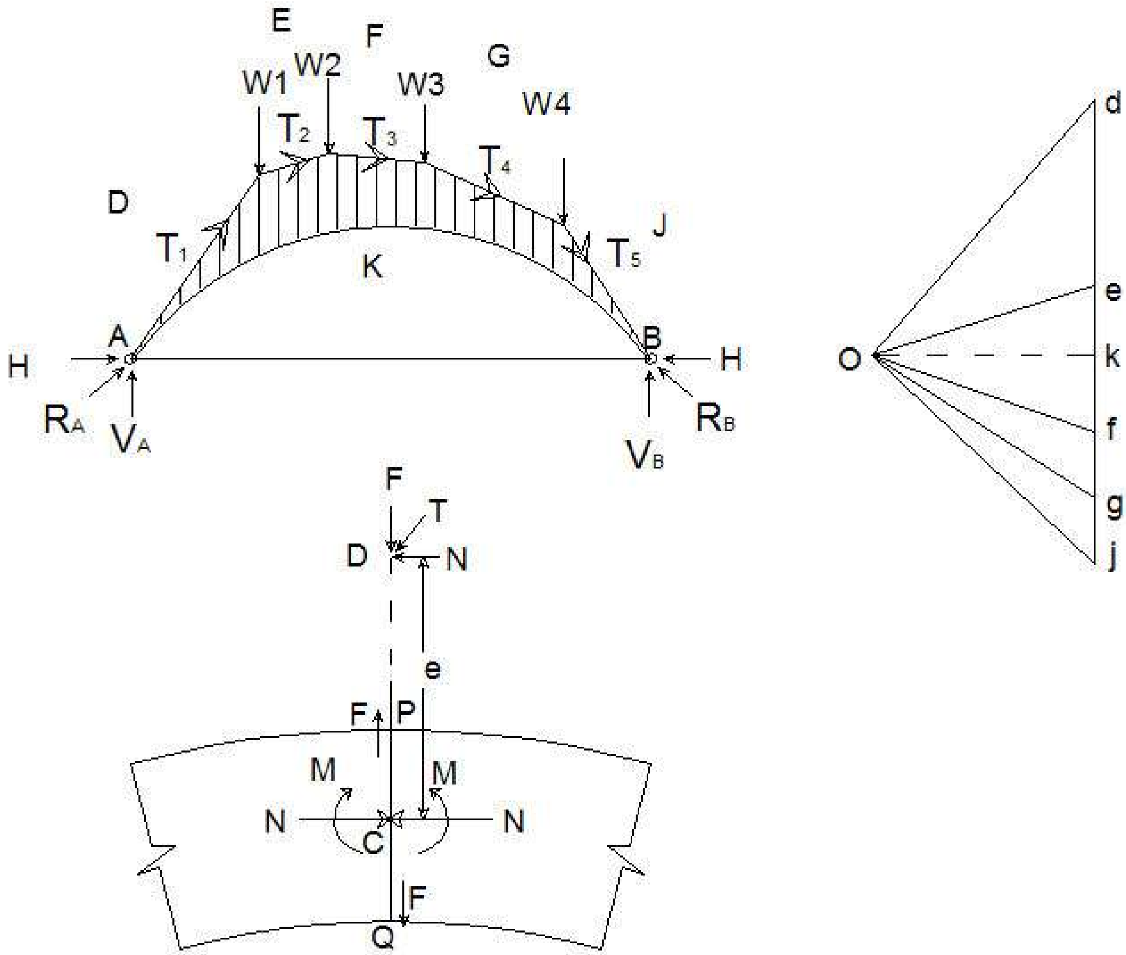
Thus the loads get transferred partly by axial compression and partly by flexure. In axial compression, at each cross section, the structure is subjected equal stresses. Reduction in bending moment results smaller and economical sections for an arch compared to a beam to transfer the same load.

Depending upon the number of hinges, arches may be divided into

a) Three hinged arch
b) Two hinged arch
c) Single hinge arch
d) Tied arch
e) Fixed arch



Linear (Theoretical) Arch and Line of Thrust:



Consider a system of jointed linkwork inverted about the support pointsA and B, with loads as shown in Fig. Under a given system of loading, each link will be in a state of compression. The magnitudes of thrusts T_1, T_2, T_3etc., can be known by the raysOd, Oe, etc., in. the force polygon drawn .Therefore a funicular polygon , which is encased in the corresponding arch with AB supports , is obtained with link work inverted.

This inverted linkwork or funicular polygon is known as “theoretical arch” or “linear arch” and also called as “line of thrust”

Actual arch: it is however not possible to construct the actual arch of shape of theoretical arch.The moving loads will change the shape of theoretical archand it cannot be made to change its shape to suit the varying load positions.Therefore in actual practice an arch is made

1) Parabolic
2) Circular and
3) Elliptical in shape

Considera cross-section PQ of the arch as shown in above. Let T be the resultant thrust acting through D along the linear arch. The thrust T is neither normal to the cross-section nor does it act through centre C of the cross-section.

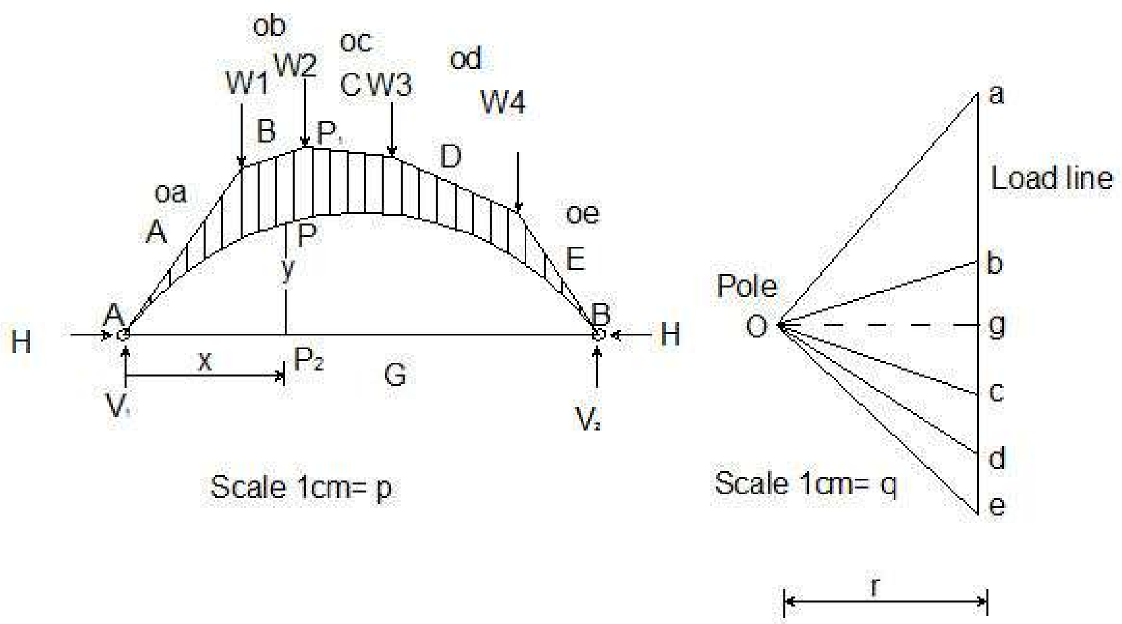
Therefore the resultant thrust T can be resolved into normal component (N)and tangentialcomponent (F)

The tangential component F willcause shear force at the section PQ.The Normal force N acts eccentrically (CD) to centre of cross section.
Thus the action of N at D can be transformed into two forces as

(i) A normal thrust N at C.
(ii) A bending moment M=N x e at C.

Where e-eccentricity CD
Hence the arch is subjected to straining actions
i) shear force also called as radial shear(F)
ii) Bendingmoment(M)
and iii) normal thrust(N)

Eddy's theorem:



Eddy's theorem states that "The bending moment at any section of an arch is proportional to the vertical intercept between the linear arch (or theoretical arch) and the centre line of the actual arch."

Consider any point on arch at a distance x from left support and y from horizontal

Let P_1, P_2 section passing through P.

H- Horizontal reaction

Funicular polygon represents the B.M diagram due to the external loads.

P_1, P_2 is the vertical interception of B.M diagram at X

Let arch is drawn to a scale 1cm=pmetre, and load diagram is plotted to a scale 1Cm =q kilometre.

If the distance of pole from load line is r cm.

Scale of B.M diagram= pqr KN-m

But theoretically, B.M at P

$M=V_1x-W_1(x-a)-Hy$

$=\mu_x-Hy$

Where μx usual B.M at section due to loading on s.sbeam .from fig,

$\mu x = P_1P_2 = P_1P_2$ X scale of B.M diagram.$=P_1P_2(pqr)$

$Hy=PP_2=PP_2$ X scale of B.M diagram.$=PP_2(pqr)$

Therefore, $M_P=\mu_x-Hy=P_1P_2(pqr)-PP_2(pqr)=PP_1(pqr)$

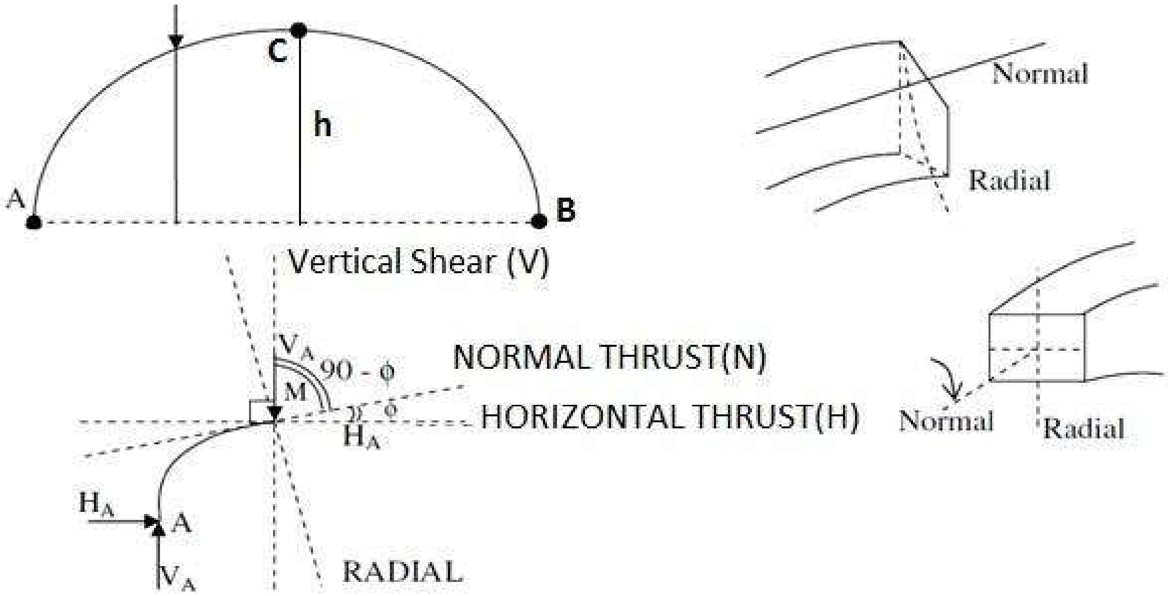
Hence the ordinate (PP_1) between linear arch and actual arch is proportional to the B.M

This proves Eddy's theorem.

Three hinged arch:

A three hinged arch is a statically determinate structure. It consists of two hinges at each abutment support called as springing and third hinge at crown.It has four reaction components.V_A&H at left hinge and V_B&H at right hinge. H being same with three available equations from static equilibrium and one additional equation i.e B.M at hinge crown is zero.Thus the value of H can be obtained easily.

Normal Thrust and Radial Shear:



Let reactions at A are H and V_A

Let reactions at B are H and V_B

Since BM at crown is zero, $M_C = V_A \times \frac{L}{2} - Hh = 0$

Let $\mu_c = V_A \times \frac{L}{2} \vee V_B \times \frac{L}{2}$

$$H = \frac{\mu_c}{h}$$

Similarly V_A and V_B can be obtained.

After obtaining V_A, V_B and H, radial shear (F) and Normal Thrust (N) can be obtained as follows

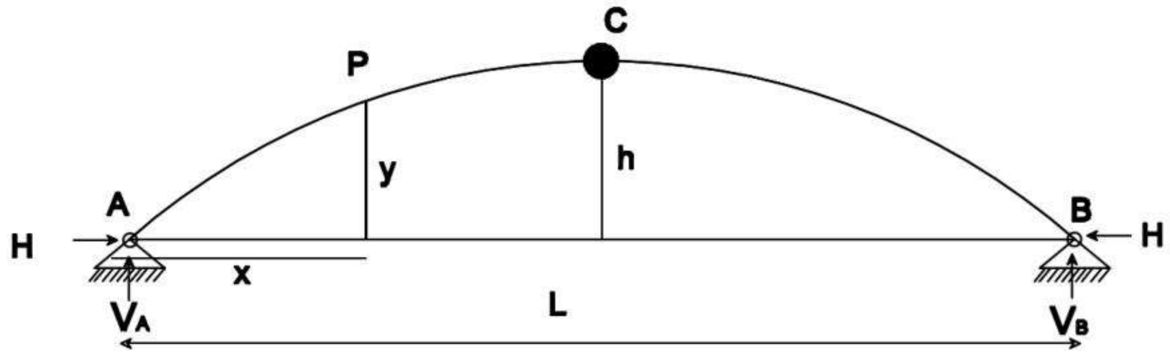
Resolving along the section,

Radial Shear (F) = HSin ϕ – VCos ϕ

Resolving normal to the section

Normal Thrust (N) = H Cos ϕ + VSin ϕ

Three hinged parabolic arch:



Equation of parabolic with respect to its span and rise can be obtained as follows.

Let y=kx(L-x) ----(1)

Where k- constant

At x=L/2, y=h central rise

Substituting in equation (1)

$$h = \frac{KL}{2}\left(L - \frac{L}{2}\right)$$

$$h = \frac{kl^2}{4}$$

Therefore,

$$k = \frac{4h}{l^2}$$

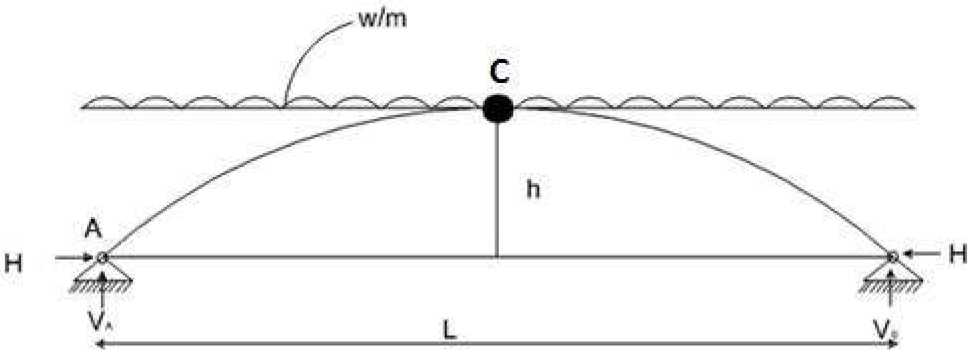
Then the equation of parabola becomes

$$y = \frac{4hx(L-x)}{l^2}$$

According to Eddy's theorem, the vertical interception between the linear arch and the center line of the actual arch gives the bending moment at a section.

When the arch is subjected to udl throughout, the Bending moment diagram would be an arch, since arch is also a parabolic arch. Hence parabolic arch will not have bending moment due to udl. It will be subjected to pure compression only.

1. A three hinged Parabolic arch of span L and rise h carries u.d.l w/m .runthroughout. Show that the horizontal thrust at each support is $\frac{WL^2}{8H}$ and also the B.M at any point of arch is zero.



Total load = WL

$$V_A = V_B = \frac{WL}{2}$$

Taking moment of all the forces about hinge A,

$$V_B L - \frac{WL^2}{2} = 0$$

Taking moment of forces left about C,

$$H_A \text{ x } h = \frac{WL}{2} \times \frac{L}{2}$$

$$H_A = \frac{WL^2}{8H}$$

$$H_A = H_B = \frac{wL^2}{8h}$$

B.M at any section x-x

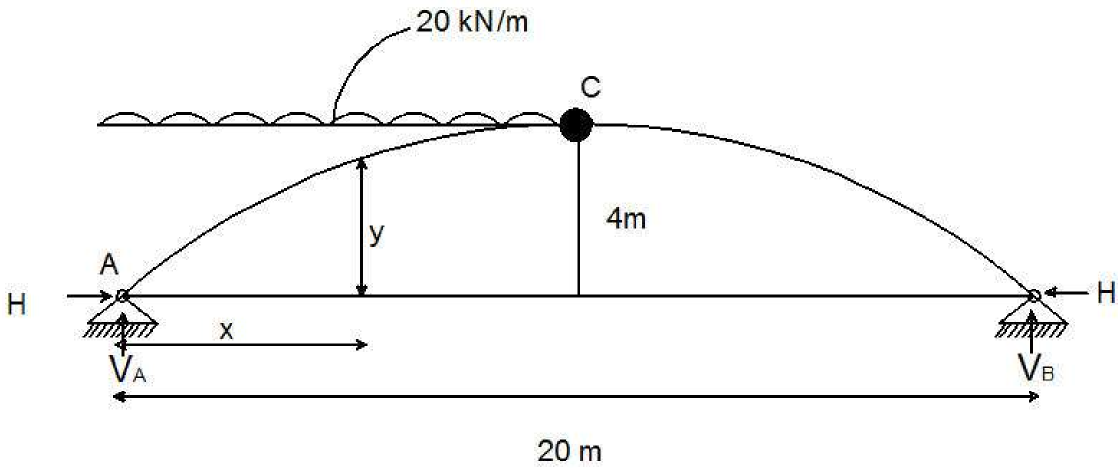
$$M_x = V_A x - \frac{wx^2}{2} - Hy$$

$$M_x = \frac{wL}{2}x - \frac{wx^2}{2} - \frac{wL^2}{8h} \times \frac{4hx(L-x)}{L^2}$$

$$M_x = \frac{wL}{2}x - \frac{wx^2}{2} - \frac{wLx}{2} + \frac{wx^2}{2}$$

= 0

2. A three hinged parabolic arch of span 20m and rise 4m carries audl of 20kN/m run on the left of the span. Find the maximum B.M for the arch.



Taking moments about A

$20V_B=(20X10)5$

$V_B=50$ kN

$V_A=150$ kN,

Taking moments about C towards right

Hx4=50x10

H=125kN

At any distance of x from A,

$y=\frac{4hx(L-x)}{L^2}=\frac{4\times4(L-x)}{400}$

$y=\frac{x(20-x)}{25}$

To obtain max B.M, $\frac{dM_x}{dx}=0$

$$M_x=V_A x-\frac{20x^2}{2}-Hy$$

$$M_x=150x-10x^2-125\times\frac{x(20-x)}{25}$$

$$M_x=50x-5x^2$$

$$\frac{dM_x}{dx}=50-10x=0$$

Hence $x = 5$m

$M_{max} = 50 \times 5 - 5 \times 5^2 = +125$ kN-m

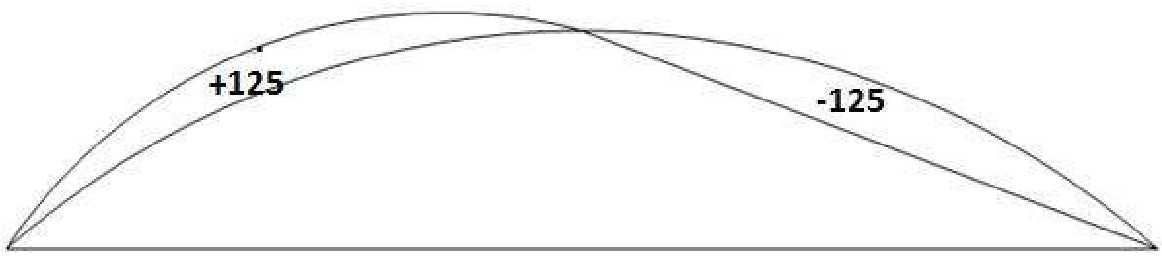
Similarly at a distance x from B,

$M_x = 5x^2 - 50x$

$\frac{dM_x}{dx} = 10x - 50 = 0$

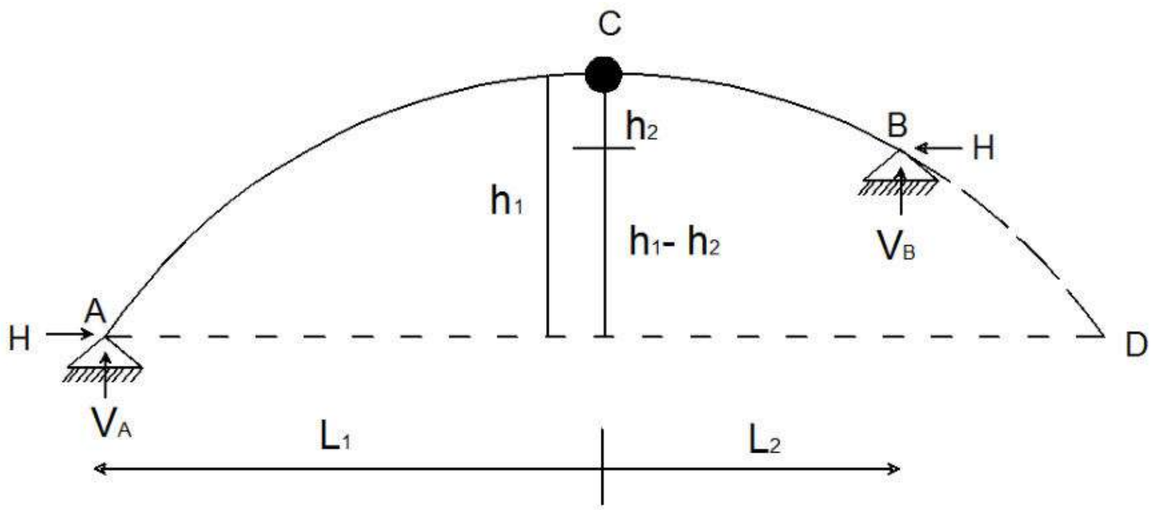
Hence $x = 5$m

$M_x = -125\, kN - m$



Bending Moment Diagram

Parabolic arch at different levels of height:



$L_1 = \frac{L\sqrt{\sqrt{h1}}}{\sqrt{h1}+\sqrt{h2}}$; $L_2 = \frac{L\sqrt{\sqrt{h2}}}{\sqrt{h1}+\sqrt{h2}}$

Let ACB is extended upto D

where L is the horizontal length between AD= $L=2L_1$

h_1- height of C above A,

h_2- height of c aboveB.

d=difference of levels

$d=h_1-h_2.$

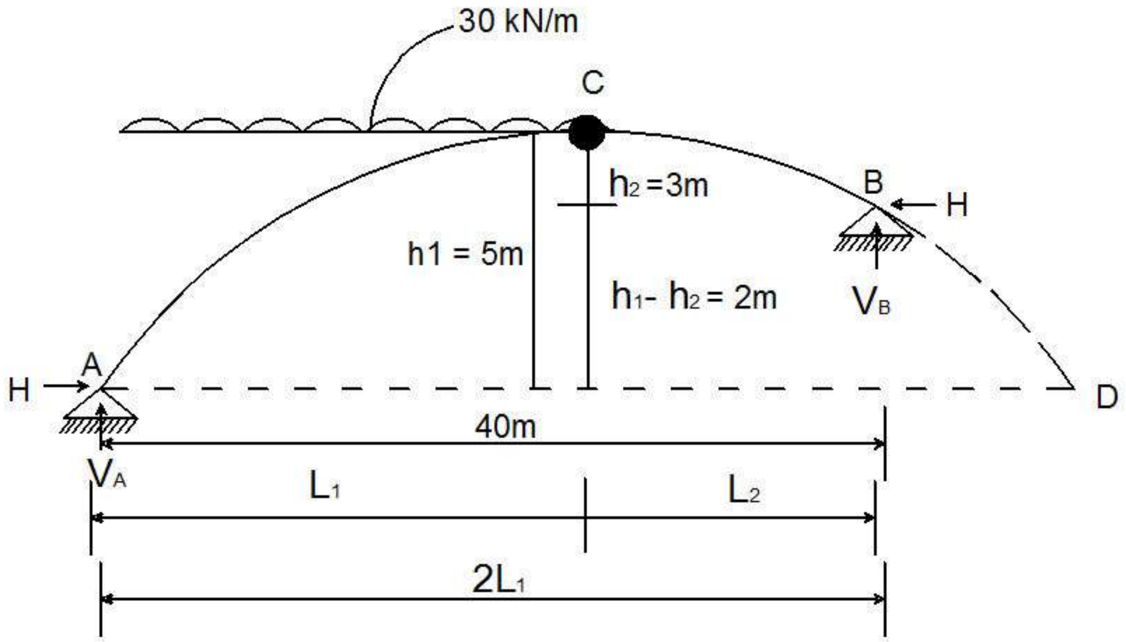
L_1= horizontal lengths betweenAC, L_2= horizontal lengths betweenCB.

L=span of arch AB.

Ordinate at any point above AD line

$$y = \frac{4h_2 x(L-x)}{L^2}$$

3. A three hinged parabolic arch having suppourts at different levels as shown in the fig.carries a udl of 30kN/m over the span left of the crown. Determine the horizontal thurst developed.also find the B.M, normal thrust and radial shear at a section 15m from left suppourts.



h_2=3m,h_1=5m

L_1 + L_2=40m

$L_1 = \frac{L\sqrt{\sqrt{h1}}}{\sqrt{h1}+\sqrt{h2}}$; $L_2 = \frac{L\sqrt{\sqrt{h2}}}{\sqrt{h1}+\sqrt{h2}}$

By substituting the values of L, h_1& $h_2$in the above formulas we get,

L_1= 22.55m, L_2=17.45m

$$y = \frac{4h_2 x(L-x)}{L^2} = \frac{20(Lx-x^2)}{L^2}$$

When x=15m , L=40m, h=3m

y=2.94m

Taking moments about C towards right,

V_BX17.45=H x 3

Therefore H=5.82V_B-----(1)

Taking moments about A,

M_A=V_Bx40+Hx2-((30x2.55^2)/2)=0

V_A= 528 kN, V_B = 147.7 kN,

Substitute the $V_B$in equation (1) we get

H=859.6KN

when x=15m,y=4.44m.

M_{15}=V_Ax15-Hx4.44-((30x15^2)/2)

=740.4kN-m

$$\frac{dy}{dx} = \tan\theta = \frac{20}{45.1\,X\,45.1}(45.1-2x)$$

When X=15m

θ =8.44^0

V_{15} = 528.8-(30X15)

= 78.8kN.

N=Vsinθ+Hcosθ

substitute H=859.6kN, θ =8.44^0,V=78.8kN.

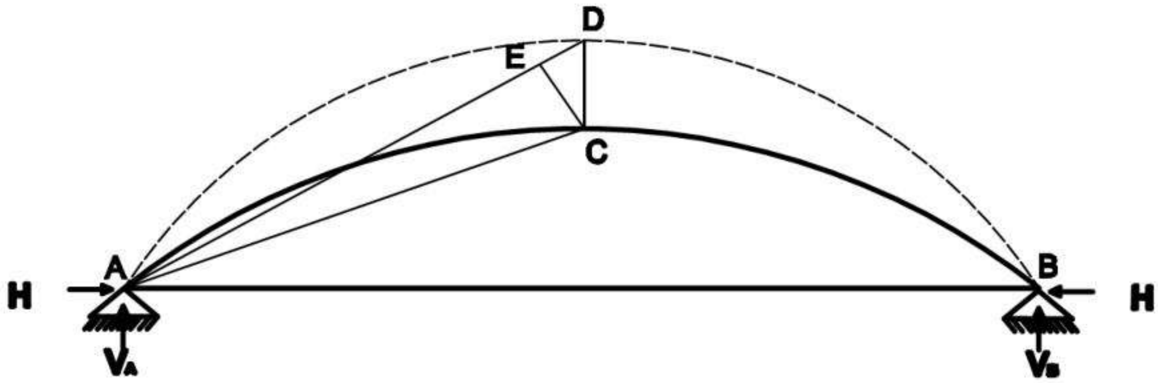
N =861.86kN

Radial shear F=Hsinθ-Vcosθ

substitute H=859.6kN, θ =8.44^0,V=78.8kN.

F=48.31kN.

Effect of change in temperature on three hinged arch:



Consider a three hinged arch ACB of span L and centaral rise h. Let t is the rise of temperature α is the co-efficient of expansion.

Let length of arch increases with rise of temperature. Since end hinges A&B cannot under go any displacement, the crown hinge c of arch will rise upwards from C to D.

Therefore increase in arch = arcAD-arc AC

Length of chord AD= length of chord AC (1+αt)

=(AC+ACαt)

Increse in length of arch =AD-AC

=(AC+ACαt)-AC

=ACαt ----(1)

Through C draw perpendicular CE on AD

Cosidering AC=AE

ED=ACαt -----(2)

CD=EDsecθ=ACαtsecθ

Considering <ADC=<ACM

CD=ACαtsecθ=ACαt$\frac{AC}{CM}$ =AC²αt/CM -----(3)

$AC^2 = AM^2 + CM^2$

= $(l/2)^2 + h^2$

= $(l^2+4h^2)/4$ ----(4)

Substitute (4)in (3)

CD=$((l^2+4h^2)/4)\frac{\alpha t}{h}$

CD =$(l^2+4h^2)\frac{\alpha t}{4h}$

There fore dh=$(l^2+4h^2)\frac{\alpha t}{4h}$

Effect of temperature rise on horizontal thrust (H):

Let h- ride of arch

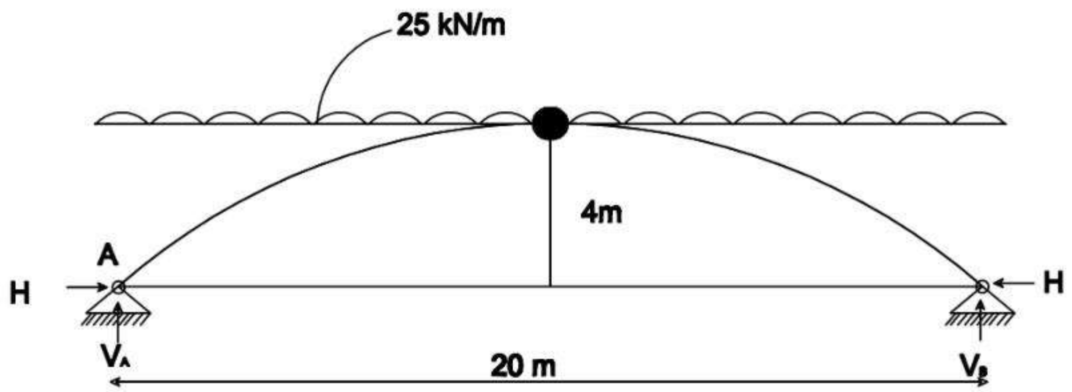
dh- increse in rise of arch or in h

dH-decrease of horizontal thrust due to increase in temperature

H- horizontal thrust

$\frac{-dH}{H} = \frac{dh}{h}$ or $dH = \frac{-dh}{h}(H)$

4. A three hinged arch of span 20m and rise 4m carries udl of 25kN/m.find the horizontal thrust for the arch. If now the arch is subjected to a rise of temperature of 40^0C,find what change in horizontal thrust will occur. Take $\alpha = 12 \times 10^{-6}$ per^{0}C.



Before rise in temperature

Taking moments about crown c,

$\frac{WL}{2}10$=Hx4+$\frac{w}{2}10^2$

Therefore H=312.5kN.

Increase in arch, dh=(l^2+4h^2)$\frac{\alpha t}{4h}$

=(20^2+4x4^2)$\frac{12x10-6}{4X4}X40$

dh=0.01392m

decrease in H=dh=$\frac{-dh}{h}$(H)

=0$\frac{-0.01392}{4}X312.5$

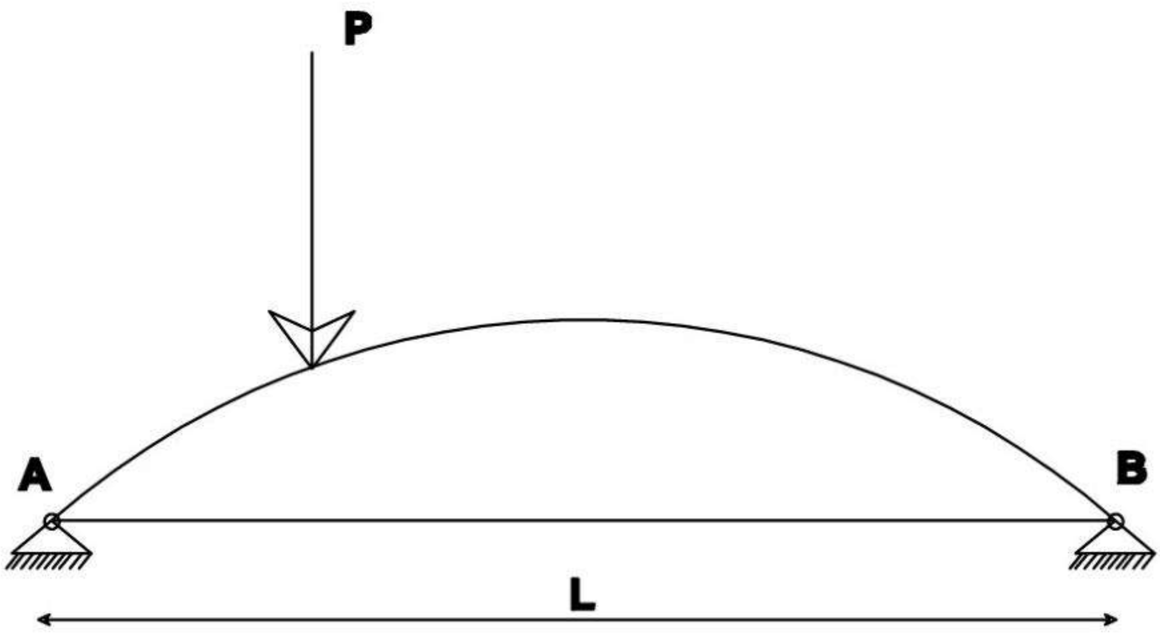
There fore dH=-1.0875 kN

Two Hinged Arches

Two-hinged arch is the statically indeterminate structure to degree one. A typical two-hinged arch has four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of static indeterminacy is one for two hinged arch.

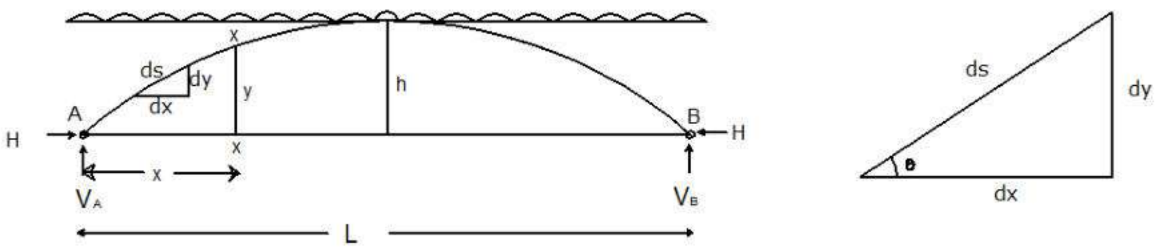
Thought three hinged arch have a hinge at the crown, it cannot move efficient than two hinged arch, but it makes the calculation simple. Two hinged arches are more practicable

Two hinged arches are parabolic or circular. In two hinged arches horizontal thrust may be taken as redundant force.



A typical two hinged arch is shown below

Determination of ‘H’ using First Theorem of “Castigliano” :



Let H – Redundant Force at B

But B is actually hinged

Bending Moment at X-X

$M_x = \mu_x - Hy$

Where μ_x - bending Moment for Simply Supported beam

We know that Strain Energy stored in arch,

$$U=\int_A^B \frac{M_x^2}{2EI}$$

Since B is hinged , there is no horizontal displacement at B

$$\frac{\partial U}{\partial H}=0$$

But

$$U=\int_A^B \frac{(\mu ¿¿ x-Hy)^2}{2EI}ds ¿$$

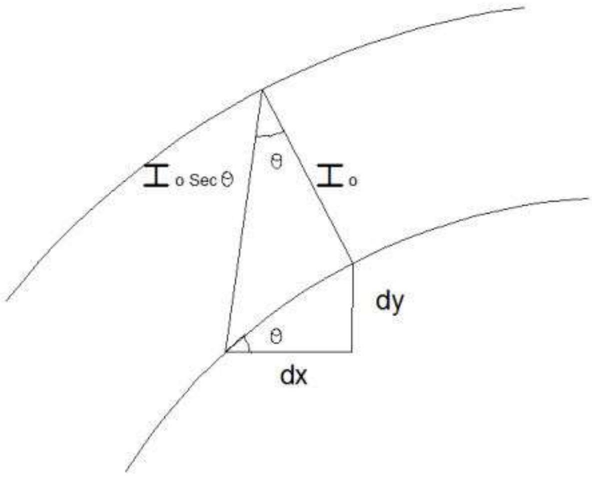
$$\frac{\partial U}{\partial H}=\int_A^B \frac{2(\mu ¿¿ x-Hy)\times(-y)}{2EI}ds=0 ¿$$

$$\frac{\partial U}{\partial H}=-\int_A^B \frac{\mu_x y}{EI}ds+\int_A^B \frac{Hy^2}{EI}=0$$

$$H=\frac{\int_A^B \frac{\mu_x y}{EI}ds}{\int_A^B \frac{y^2}{EI}ds}$$

But ds = $\sqrt{(dx^2 ¿+dy^2) ¿}$

For Simplicity, I varies as I= $I_0 sec\theta$



Where $I_0 = Moment\ of\ Inertia\ at\ crown$

Also $ds=dx\,sec\theta$

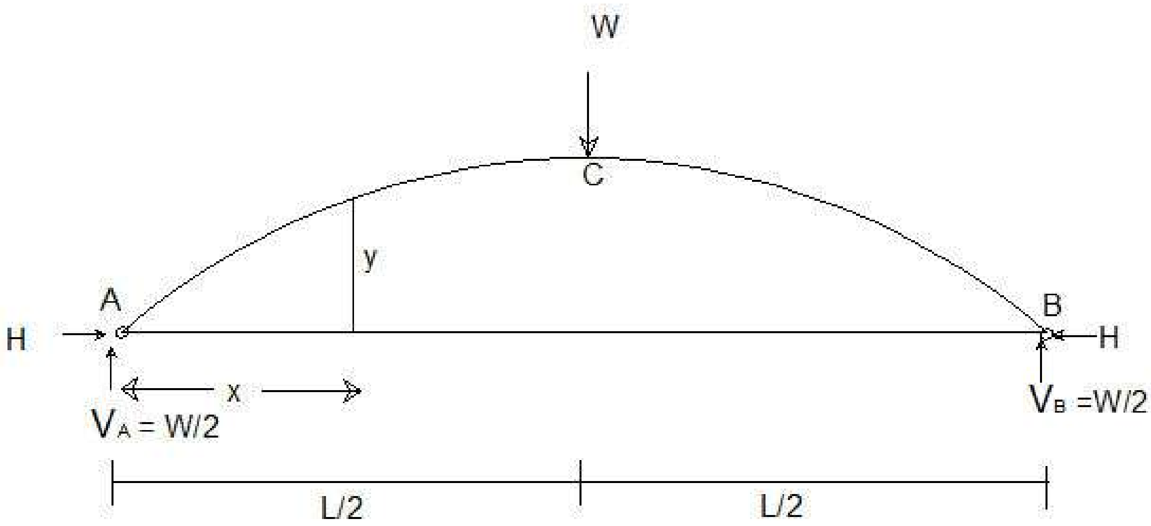
Substituting in equation ‘H’

$$H=\frac{\int_0^l \frac{(\mu ¿¿ x\, y)\, dx sec\theta}{E I_0 sec\theta}}{\int_0^l \frac{y^2\, dx sec\theta}{E I_0 sec\theta}} ¿$$

$$H=\frac{\int_0^l \mu_x\, y\, dx}{\int_0^l y^2 dx}$$

Analysis of Two hinged Parabolic Arches:

Ex: A two hinged parabolic arch of span L and rise ‘h’ carries a concentrated load W at the crown. Determine the expression for horizontal thrust H developed at Springing.



$V_A = V_B = \frac{W}{2}$ (due to symmetry)

Take a Section X-X

$\mu_x = \frac{W}{2} x$

$$y = \frac{4hx(L-x)}{l^2}$$

$\int_0^l \mu_x y\,dx = \int \frac{W}{2} x \times \frac{4hx(L-x)}{l^2}$

$\int_0^l \mu_x y\,dx = \frac{2Wh}{l^2} \times 2 \int_0^{l/2}$ ¿¿) dx

$= \frac{4wh}{l^2}$¿] over 0 –L/2

$= \frac{5Whl^2}{48}$

Denominator

$\int_0^l y^2\,dx = \int_0^l \left[\frac{4hx(L-x)}{l^2}\right]^2 dx$

¿ $\frac{16h^2}{l^4} \times 2 \int_0^{l/2} [l^2 x^2$¿$-2lx^3 + x^4]\,dx$ ¿

¿ $\frac{32h^2}{l^4}\left[l^2 \frac{x^3}{3} - 2l\frac{x^4}{4} + \frac{x^5}{5}\right]_0^{l/2}$

$$¿ \frac{32h^2}{l^4}\left[\frac{l^5}{24}-\frac{l^5}{32}+\frac{l^5}{160}\right]$$

$$¿ \frac{8}{15}h^2 l$$

Therefore

Horizontal Thrust (H) = $\dfrac{\dfrac{5Whl^2}{48}}{\dfrac{8}{15}h^2 l} = \dfrac{25}{128}\dfrac{Wl}{h}$

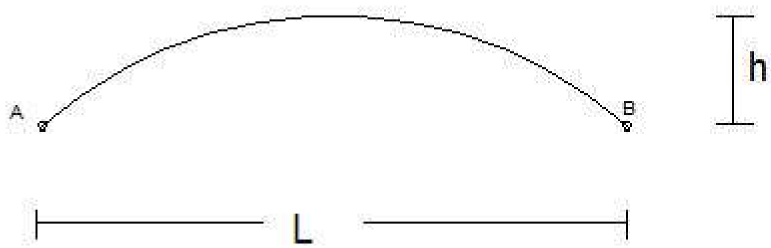
Effect of Temperature on Two hinged Arch:

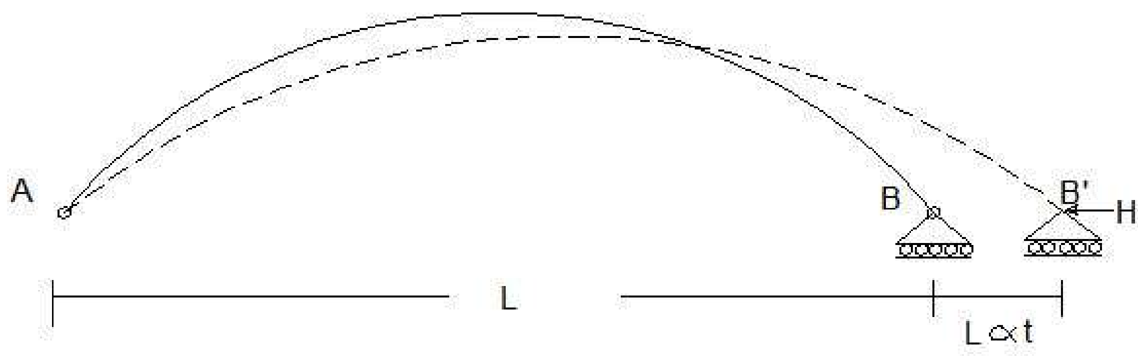
Consider two hinged arch subjected to rise in temperature by ‘t’ degrees. If B hinge is replaced by a roller as shown in figure.

Let $\alpha = Coefficient\ of\ thermal\ expansion$

H = Horizontal Thrust required to bring back B to B which infers that H is the horizontal Thrust developed in the two hinged arch using the theorem of Castigliano.

Change in Displacement = $\dfrac{dU}{dH} = L\alpha t$





In this case M = -Hy

$$\frac{dM}{dH} = -y$$

$$\frac{dU}{dH} = L\alpha t$$

$$\int \frac{d}{dH}\frac{(M^2)}{2EI} ds = L\alpha t$$

$$\int \frac{M}{EI}\frac{dM}{dH} ds = L\alpha t$$

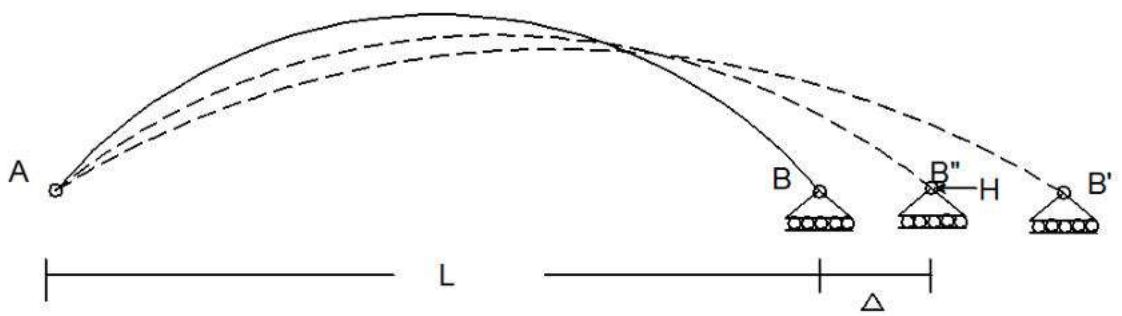
$$\int (-Hy)(-y)\frac{ds}{EI} = L\alpha t$$

$$H = \frac{L\alpha t}{\int y^2 \frac{ds}{EI}}$$

The above expression is Horizontal Thrust due to temperature change only.
If it is confined with loading and settlement of support then total horizontal thrust

$$H = \frac{\int \frac{\mu_x y}{EI} ds + L\alpha t - \Delta}{\int y^2 \frac{ds}{EI}}$$

Effect of yielding of Supports:



Let B hinge is replaced by roller
Let due to loads, the support moves to point B
Let $\mu - Moment\ due ¿ simply\ supported\ case$
$y - Moment\ due ¿ horizontal\ force$

BB = $\int \frac{\mu_x y}{EI} ds$

If H- Horizontal force at springing level
M –Moment due to H = Hy

$¿ \int \frac{\mu_x y}{EI} ds$

$$¿\int \frac{(Hy)\,y}{EI} ds$$

$$¿\int \frac{H\,y^2}{EI} ds$$

Since the support is yielding by Δ from B to B

$$\int \frac{H\,y^2}{EI} ds = B\,B - \Delta$$

$$H = \frac{\int \frac{\mu_x\,y}{EI} ds - \Delta}{\int y^2 \frac{ds}{EI}}$$

If the support is elastic and if yields by k due to unit horizontal force at support then $\Delta = kH$

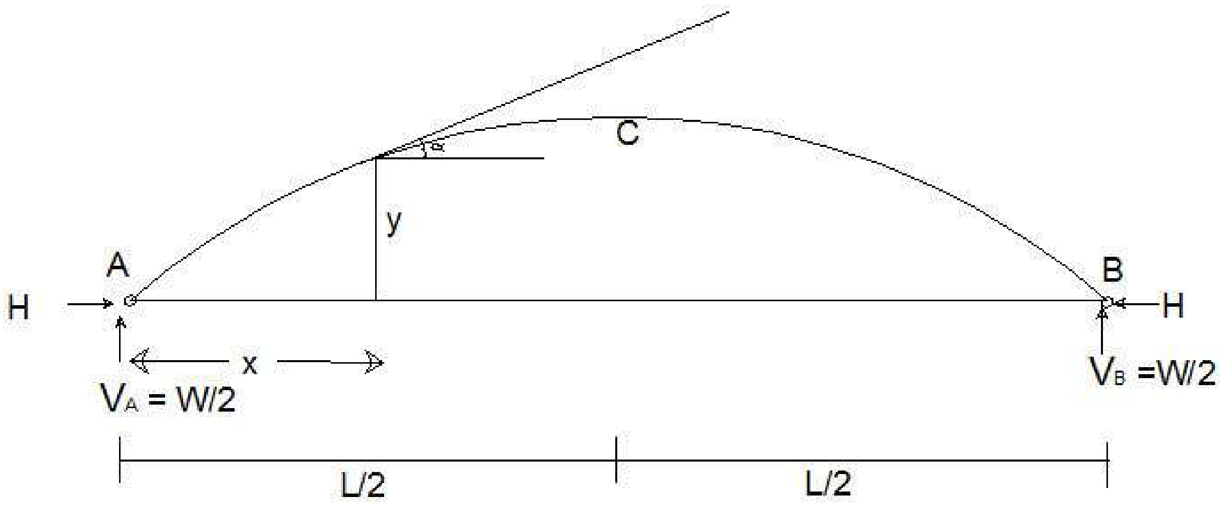
$$H \int \frac{y^2}{EI} ds = B\,B - \Delta$$

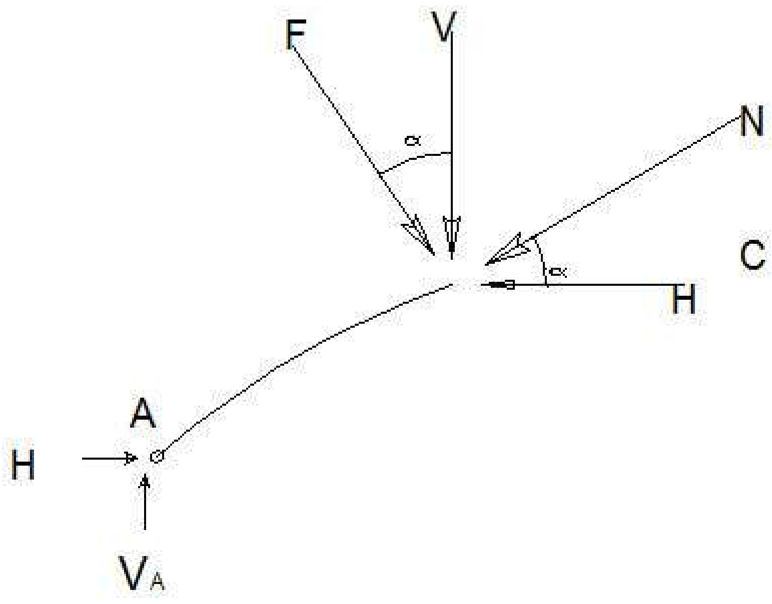
$$H \int \frac{y^2}{EI} ds = \int \frac{\mu_x\,y}{EI} ds - kH$$

$$H\left(\int \frac{y^2}{EI} ds + k\right) = \int \frac{\mu_x\,y}{EI} ds$$

$$H = \frac{\int \frac{\mu_x\,y}{EI} ds}{\int \frac{y^2}{EI} ds + k}$$

Effect of Shortening of Rib:





The cross section of the arch is subjected to normal thrust also. The arch being made up of elastic material , counters shortening of the rib. This shortening reduces the horizontal thrust developed.

Expression for horizontal thrust due to rib shortening:

From figure N = VSinα + HCos α

M = $\mu - Hy$

Where V = Beam Shear

Starain Energy U = $\int \frac{M^2}{2EI} ds + \int \frac{N^2}{2EA} ds$

Where A –Area of Crossection at section X-X

If the support is unyielding, the horizontal displacement of arch is Zero.

$\frac{dU}{dH} = 0$

$$\int \frac{M}{EI} \frac{dM}{dH} ds + \int \frac{N}{EA} \frac{dN}{dH} ds = 0$$

$$\text{But } \frac{dM}{dH} = \frac{-y \wedge dN}{dH} = cos\alpha$$

$$\int \frac{(\mu - Hy)(-y)}{EI} ds + \int \frac{(VSin\alpha + HCos\alpha)(\cos\alpha)}{EA} ds = 0$$

$$-\int \frac{\mu y}{EI} ds + H \int \frac{y^2}{EI} ds + \int \frac{VSin\alpha\, Cos\alpha}{EA} ds + H \int \frac{\cos^2 \alpha}{EA} ds = 0$$

$$H = \frac{\int \frac{\mu y}{EI} ds - \int \frac{VSin\alpha Cos\alpha}{EA} ds}{\int \frac{y^2}{EI} ds + \int \frac{\cos^2 \alpha}{EA} ds}$$

Neglecting effect of shear

$$H = \frac{\int \frac{\mu y}{EI} ds}{\int \frac{y^2}{EI} ds + \int \frac{\cos^2 \alpha}{EA} ds}$$

Considering the term $\int \frac{\cos^2 \alpha}{EA} ds$ at crown and springings,it has some definite value.

Usually Crossection areat at crown is small and at springing is large

Hence $\frac{A}{Cos\alpha}$ = Constant = A_m ASecα

$$\int \frac{\cos^2 \alpha}{EA} ds \; becomes \int \frac{Cos\alpha}{E A_m}$$

$$\int \frac{\cos^2 \alpha}{EA} ds = \int_0^L \frac{1}{E A_m} dx = \frac{L}{E A_m}$$

Substitute in Expression H

$$H = \frac{\int \frac{\mu y}{EI} ds}{\int \frac{y^2}{EI} ds + \frac{L}{E A_m}}$$

$$H = \frac{\int \frac{\mu_x y}{EI} ds - \Delta}{\int y^2 \frac{ds}{EI} - \frac{L}{E A_m}}$$

Where $\Delta = kH$

Theory of Structures

III year-I Semester

<u>Unit-II</u>

<u>Influence Line Diagrams and Moving Loads</u>

Learning Material

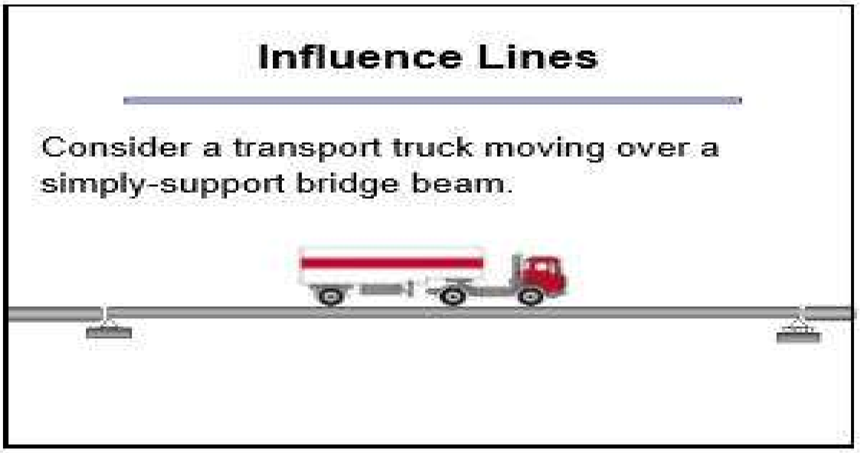
INTRODUCTION:

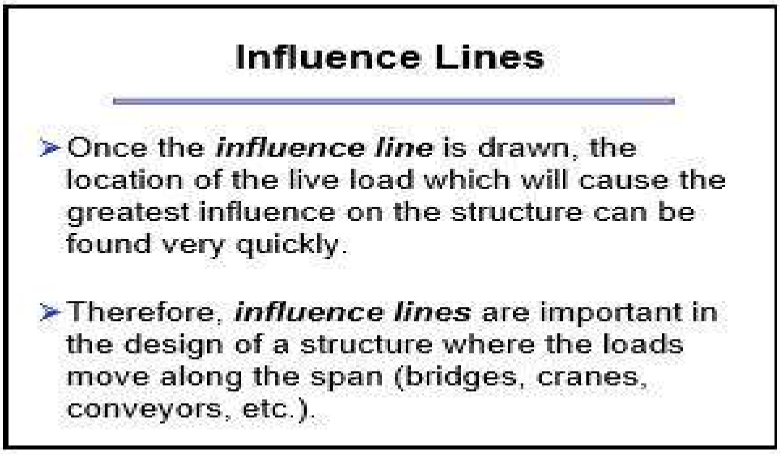
Influence lines are important in the design of structures that resist large live loads.

If a structure is subjected to a live or moving load, the variation in shear and moment is best described using influence lines.

Although the procedure for constructing an influence line is rather simple, it is important to remember the difference between constructing an influence line and constructing a shear or moment diagram.

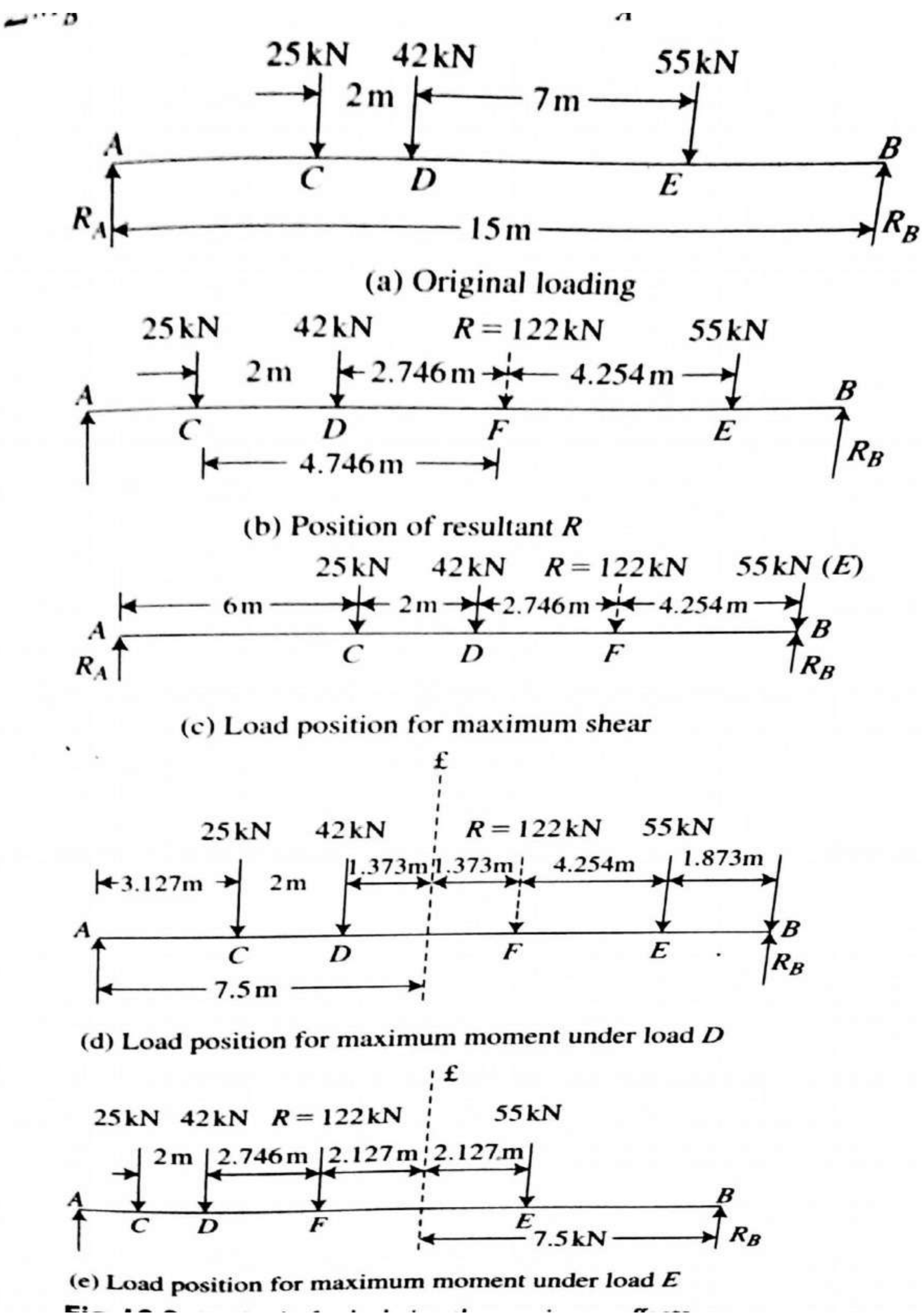
Definition: An **influence line** for a given function, such as a reaction, axial force, shear force, or bending moment, is a graph that shows the variation of that function at any given point on a structure due to the application of a unit load at any point on the structure.





MAXIMUMSHEAR FORCE AND BENDING MOMENT AT A GIVEN SECTION AND ABSOLUTE MAXIMUM SHEAR FORCE AND BENDING MOMENT:

1. A simple beam with a system of moving concentrated loads is shown in figure. Calculate the absolute maximum B.M and S.F.



(a) Original loading

(b) Position of resultant R

(c) Load position for maximum shear

(d) Load position for maximum moment under load D

(e) Load position for maximum moment under load E

The resultant of 3 loads R=25+42+55=122 kN

The location of resultant "R" is determined by taking moments of the load 'C' and dividing by the total loads,

$\bar{x}$= (42×2+55×9)/122=4.746m

Absolute maximum shear force:

The absolute maximum shear force occurs the loads are moved to the right so that largest load of 55kn is on the right support at B

$\Sigma M_A = 0; R_B = 87.4kN$

When the load (55 kN) is on support B, then

$\Sigma M_A = 0; R_B = 87.4kN$

When the load (42 kN) is on support B, then

$\Sigma M_A = 0; R_B = 63.67kN$

When the load (25 kN) is on support B, then

$\Sigma M_A = 0; R_B = 25kN$

When the load D on 'A'

It is quiet clear that neither load C on support A nor load D on support A or B gives rise to the condition of maximum S.F. only load E on support B leads to a condition of max. S.F and absolute max. S.F.

Absolute maximum bending moment:

The max. B.M should be found either under load D or under load E.

The maximum moment occures are determined by 2 variables namely 1)nearness to the resultant R   2)magnitude of the load

$\Sigma M_B = 0; R_A = 49.83kN$

$\Sigma M_{D(L.O.S)} = 255.48$ kN-m

Now for E:-

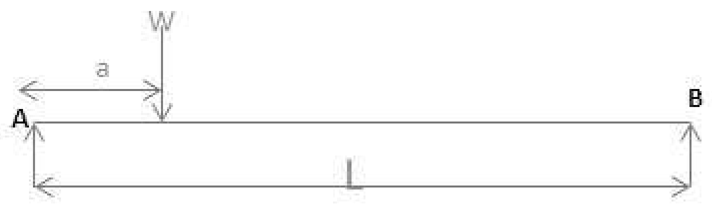
$\Sigma M_A = 0; R_B = 43.70kN$

$\Sigma M_{E(R.O.S)} = 234.80$ kN-m

Absolute maximum B.M= M_D=255.48 kN-m

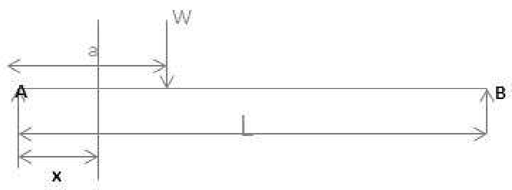
SINGLE CONCENTRATED LOAD:

1. Determine the maximum positive and negative S.F. and B.M at a section a m in a simple beam of span L when a concentrated load of W kN rolls across the beam. Also calculate the absolute S.F and B.M.



Cal of max + S.F.

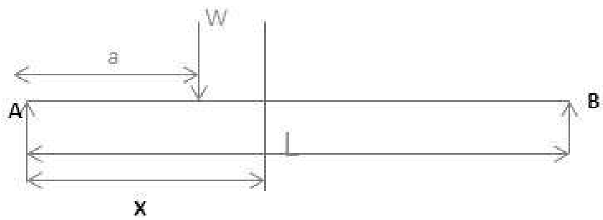
max + S.F. occurs when the load is in section xB i.e.no load xA



Shear force at X=+R_A
$\sum M_B=0$, $R_A=W(L-a)/L$
Moving from zero m at left support and L m from right support
S.F. when a=0 i.e. S.F. at A=+W
When a=X i.e. $S.F_X=W(L-X)/L$
When a=L i.e. $S.F_B=0$

+ shear force diagram

Cal of max - S.F.

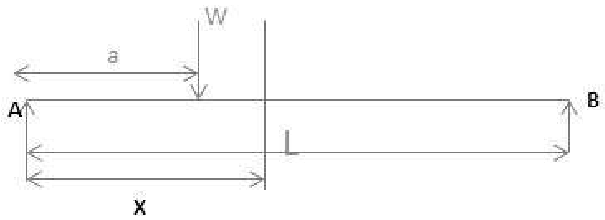


This negative S.F. occurs when the load is on section AX i.e. there is no load on XB
Shear force at X=-R_B
$\sum M_A=0$, $R_B=Wa/L$
when a=0 i.e. S.F. at A=0
When a=X i.e. $S.F_X=-WX/L$
When a=L i.e. $S.F_B=-W$

- shear force diagram

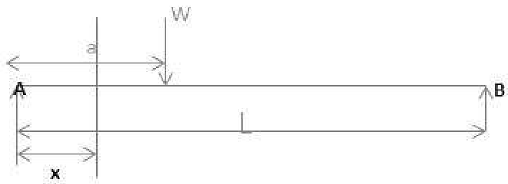
Cal of Bending Moment for the moving load W

Case i)

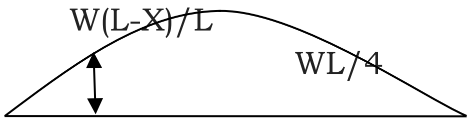


B.M_X=WX(L-X)/L
At X=0, B.M_A=0
At X=L, B.M_B=0

Case ii)



B.M_X=WX(L-X)/L
At X=0, B.M_A=0
At X=L, B.M_B=0



Shape of B.M.D

Absolute Shear Force
It is nothing but S.F_{max} of Maximum

Absolute positive shear force=+W
Absolute negative shear force=-W

UDL LONGER THAN THE SPAN:

1. A UDL of intensity 12kN/m and length more than 7m moves across a girder of span of 7m.find maximum '+'ve and '-'ve S.F. at a section 3m from left support as well as its absolute value. Similarly, determine the maximum. B.M. at the same section and the absolute value.

Max.'-'ve S.F=-7.71 kN

Max.’+‘ve S.F=13.71 kN

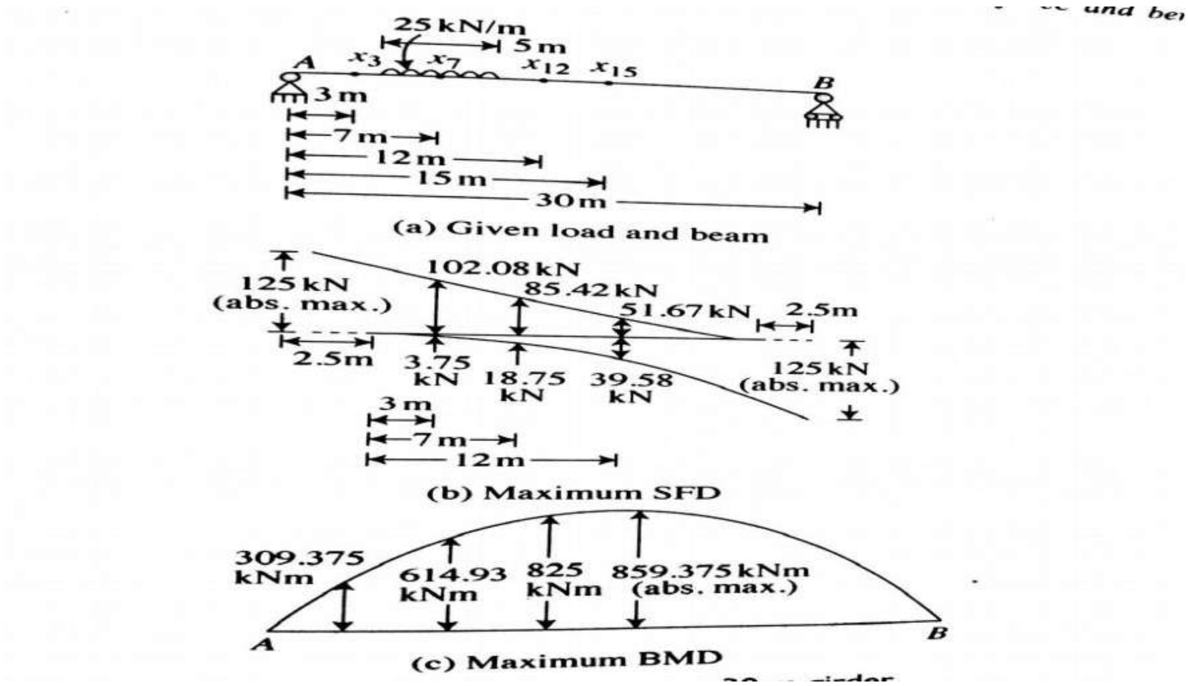
Absolute max.S.F=42 kN

Max.B.M=72 kN-m

ABSOLUTE Max.B.M=73.5 kN-m

UDL SHORTER THAN THE SPAN:

1. A UDL of length 5m and intensity 25kN/m moves across a simple beam of span 30m.determine max.’-‘ve and max.’+’ve S.F. and max.B.M at sections 3m,7m,12m from the left support and also the absolute max.S.F and B.M.draw S.F.D and B.M.D.



(a) Given load and beam

(b) Maximum SFD

(c) Maximum BMD

Calculation of max.S.F.at 3m:

Max.’-‘ve S.F:
For getting max.’-‘ve S.F the head of the UDL must be at the section
Max.’-‘ve S.F=-R_B=-3.75kN
Max.’+‘ve S.F:
For getting max.’+‘ve S.F the tail of the UDL must be at the section
Max.’+‘ve S.F=R_A=102.08kN

Max.B.M:

The condition for max.B.M when UDL shorter than the span is the section should divide the UDL in the same ratio as it divides the span.

At a section 3m M_{max}=309.375kN-m

Calculation of max.S.F.at 7m:

Max.'-'ve S.F:

For getting max.'-'ve S.F the head of the UDL must be at the section

Max.'-'ve S.F=-R_B=-18.75kN

Max.'+'ve S.F:

For getting max.'+'ve S.F the tail of the UDL must be at the section

Max.'+'ve S.F=R_A=85.42kN

Max.B.M:

The condition for max.B.M when UDL shorter than the span is the section should divide the UDL in the same ratio as it divides the span.

At a section 7m M_{max}=614.93kN-m

Calculation of max.S.F.at 12m:

Max.'-'ve S.F:

For getting max.'-'ve S.F the head of the UDL must be at the section

Max.'-'ve S.F=-R_B=-39.58kN

Max.'+'ve S.F:

For getting max.'+'ve S.F the tail of the UDL must be at the section

Max.'+'ve S.F=R_A=51.67kN

Max.B.M:

The condition for max.B.M when UDL shorter than the span is the section should divide the UDL in the same ratio as it divides the span.

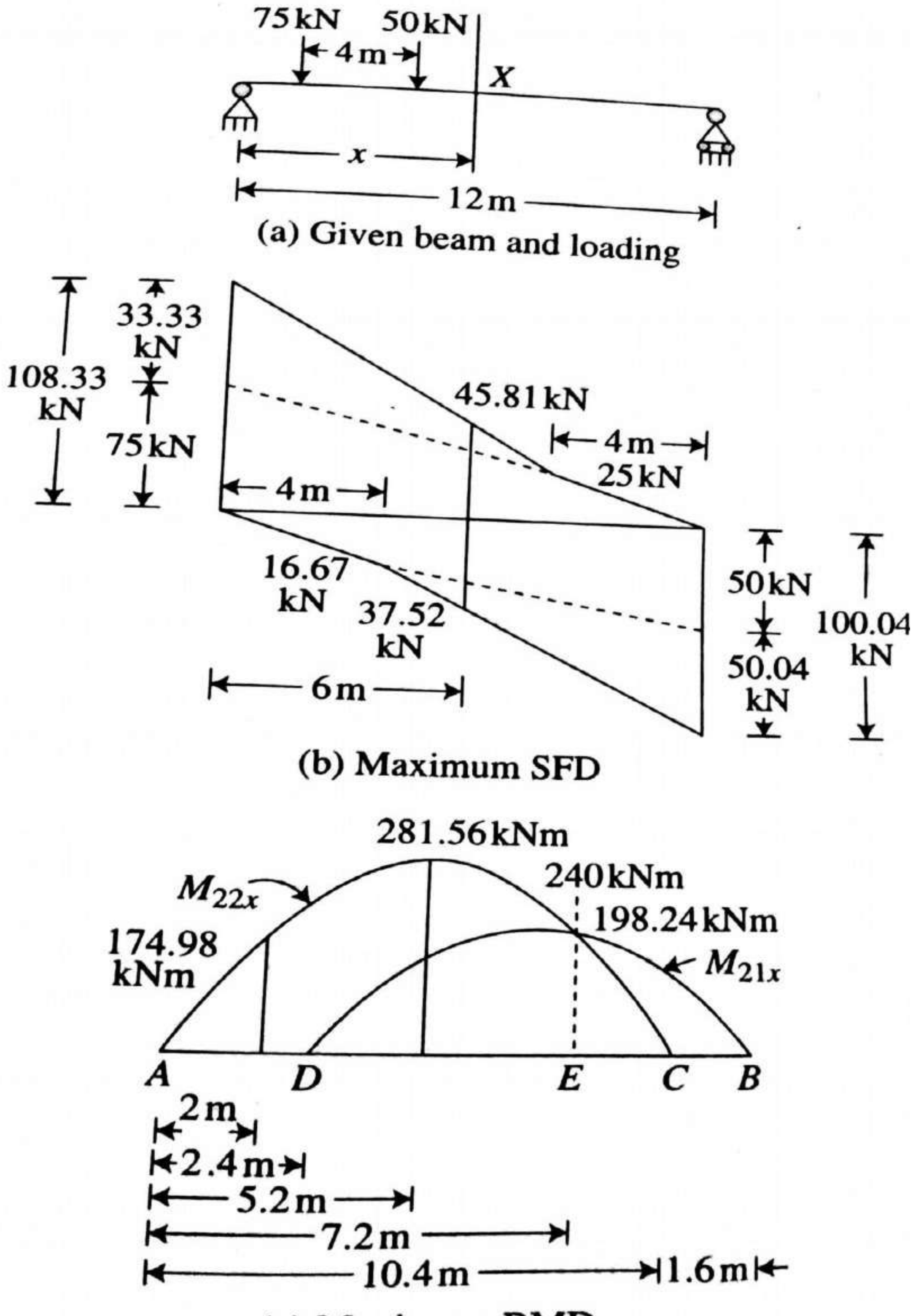
At a section 12m M_{max}=825kN-m

Absolute max.S.F=125Kn

Absolute B.M=859.375kN-m

TWO CONCENTRATED LOADS SEPERATED BYA DISTANCE:

1. Two concentrated loads of 50Kn and 75kN separated by 4m rolls across a beam of 12m span from left to right with 50kN load leading the train. Draw the max.S.F.D and B.M.D. Also, locate the position and calculate the magnitude of the absolute max.B.M.



(a) Given beam and loading

(b) Maximum SFD

(c) Maximum BMD

W_1=50kN,w_2=75kN,d=4m,l=12m

<u>Max.shear diagram</u>:

Consider any section x distance x from left end A of the girder.

When $x<d$

Negative shear $=-R_B=-50x/l$

At x=0,negative shear=0

At x=4m,negative shear=-16.67kN

When $x>d$

$R_B=(50x+75(x-4))/12$

At x=4m, R_B=-16.68kN

At x=6m, R_B=-37.52kN

At x=12m, R_B=-100.04kN

The absolute max.'-'ve S.F is 100.04kN and it occurs at support B.

<u>Max.'+'ve shear diagram</u>

$R_A=(50(l-x-4)+75(l-x))/12$

At x=0; R_A=108.33kN

At x=6m; R_A=45.81kN

At x=8m; R_A=24.97kN

When $(l-x)<4m$

$R_A=75(l-x)/12$

At x=8m; R_A=25kN

At x=12m; R_A=0

The absolute max.'+'ve S.F.=108.33kN

<u>Maximum bending moment:</u>

When $x<d$

$R_A=50(l-x)/12$

The bending moment at x

$M_{x(L.O.S)}=50x(l-x)/12$

At x=0; M_x =0

At x=4m; M_x =33.33kN-m

When $x>d$

$R_A=(75(4+l-x)+50(l-x))/12$

$M_x= R_Ax-(75\times4)$

At x=4m; M_x =133.33kN-m

At x=12m; M_x =0

When $x<(l-d)$

$R_A=(75(l-x)+50(l-x-4))/12$

$M_x= R_Ax$

At x=0; M_x =0

At x=(l-d); M_x =200kN-m

When $x>(l-d)$

$R_A=(75(l-x))/12$

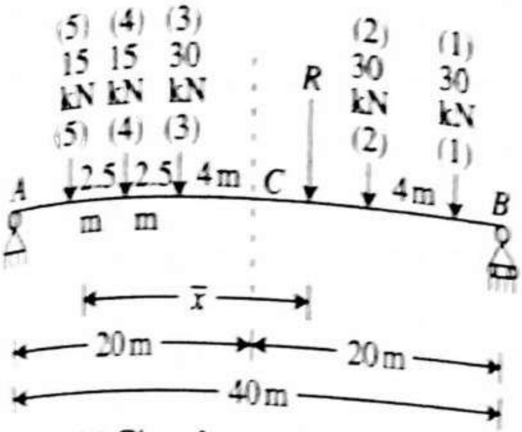
$M_x= R_Ax$

At x=(l-d); M_x =200kN-m

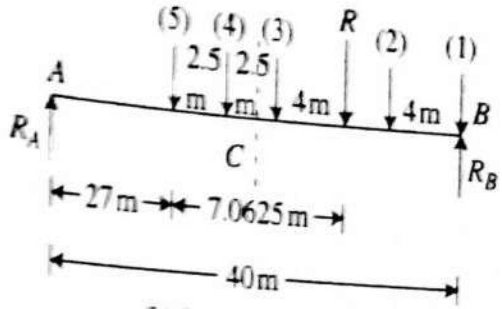
At x=12m; M_x =0

SIMPLE BEAM WITH SEVERAL CONCENTRATED LOADS:

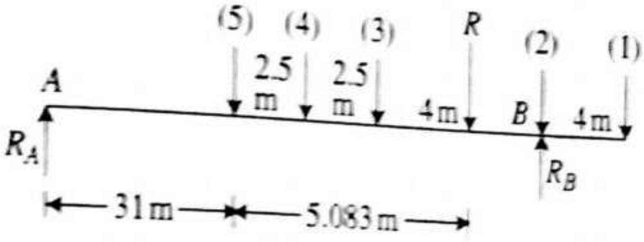
1. Determine the max.S.F and B.M in the span of a simple beam witha system of moving loads shown in figure.



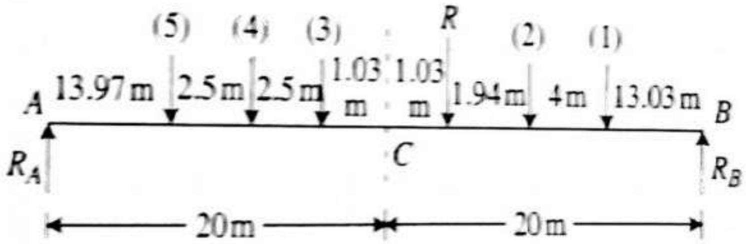
(a) Given beam with loading



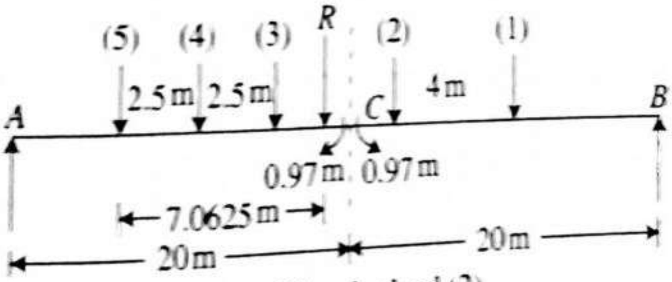
(b) Load (1) on supports



(c) Load (2) on support B



(d) Maximum BM under load (3)



(e) BM under load (2)

Max.S.F:

R=resultant of all loads=15+15+30+30+30=120kN

Resultant of all loads lies at a distance of $\bar{x}$ from load 5.

$\bar{x}$=7.0625m

when 1st load is placed on B

R_B=102.19kN

The next load i.e 2nd load is placed on B

R_B=81.19kN

It is quite obvious that S.F. decreases

Therefore max. S.F occurs when load 1 in on support B shown in fig.

Then max .S.F.=102.19kN

Max. B.M:

We consider load 3

The distance between load 3 and resultant equi-distances of 2.0625/2=1.03m from the centre of the beam shown in fig.

R_A=56.91kN

B.M under the load 3 is

M=967.08kN-m

We consider load 2

The distance between load2 and resultant is 1.9375m

We place the load 2 on either side of centre beam

R_B=57.09kN

B.M under load 2 is

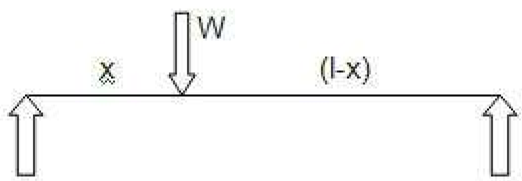
$M_{(R.O.S)}$=966.42kN-m

Max. B.M occurs at load 3=967.08kN-m

EQUIVALENT UDL:

A given system of point loads can also be converted into a system of UD static load covering the entire span such that moment induced by the static loading is equal to or greater than the moments obtained under rolling loads. Such a static loading is called **equivalent udl.**

1. Determine the equivalent UDL of teh single point load case.



Maximum B.M. at x at distance x from support A is given by

M_{MAX} =Wx(l-x)/l……………….1

If we take the intensity of the equivalent UDL over the whole span as W_{eq}, then at section x,the B.M. is

$M= (W_{eq}l)x/2-(W_{eq}l)x^2/2=W_{eq}x(l-x)/2$.....................2

Equating 1 and 2 eq's we get

$W_{eq}=2W/l$

FOCAL LENGTH:

1.A span of girder is 40m its dead load is 30kn/m. from the consideration of shear the equivalent UDL is 75kn/m. Compute the focal length of a girder.

Cal. of reactions:

$R_A = R_B = 600kN$

The dead load shear at a section x distance x from left support is given by

$V_{DL}=(30\times40)/2-30x$

Equivalent UDL shear

$V_{LL}=-0.9375x^2$

$V_{DL} + V_{LL}=0$

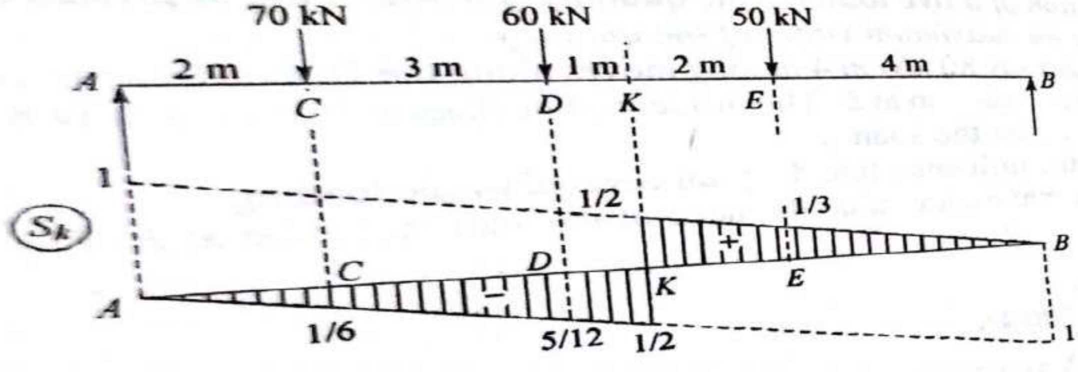
x=13.93m

the focal length=40-2×13.93=12.14m

INFLUENCE LINE DIAGRAM FOR S.F. AT AGIVEN SECTION:

1.find the shear force at the section K for the loaded girder by the method of influence lines.

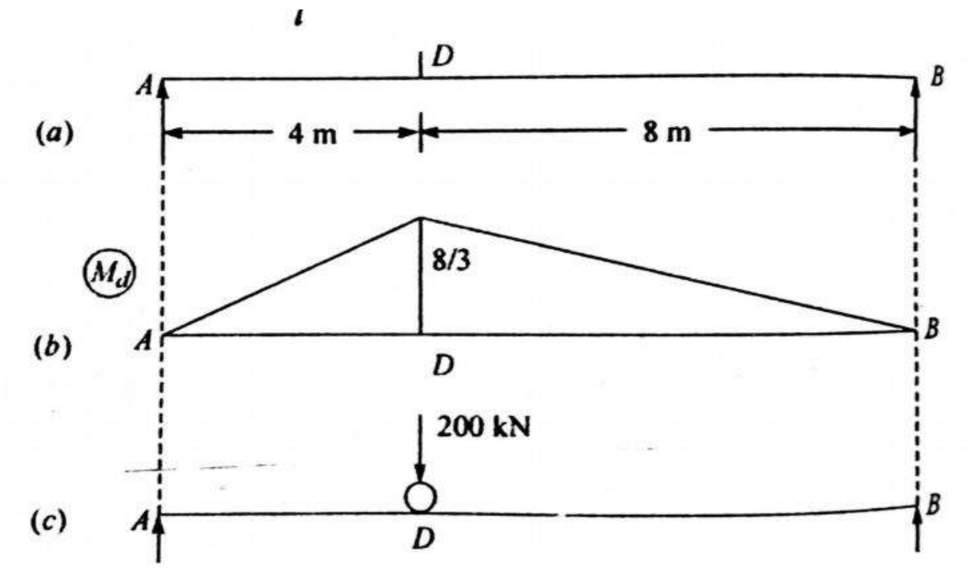
Draw the influence line diagram for the S.F. at K



S.F. at K =70(-1/6)+60(-5/12)+50(1/3)=-20kN.

INFLUENCE LINE DIAGRAM FOR B.M. AT AGIVEN SECTION:

1.a simply supported girder has a span of 12m a 200kN wheel load moves from one end to the other end on the span of a girder. Find the max.B.M. which can occur at a section 4m from the left end.



Let us first draw the influence line diagram for the bending moment at the given section D

Height of I.L.D=a(l-a)/l=8/3

By studying the influence line diagram, it is obvious that, in order the bending moment at D may be maximum; the wheel load should be placed exactly at D.

Max. B.M at D=200×8/3=533.33kN-m

LOAD POSITION FOR MAX.S.F AND B.M. AT A SECTION:

1.Find the max .'-'ve and '+'ve S.F and max. B.M. at a section 3.5m from left support ina simple beam of span 6m.a single point load of 100kN rolls across the beam.

Here a=3.5m;b=2.5m;l=6m;W=100Kn

When the load just to the left of the section we get max .'-ve- S.F.=-58.33kN

When the load is just to the right of the section we get max. Positive S.F. =41.67kN

When the load is on the section we get max.B.M=145.83kN-m

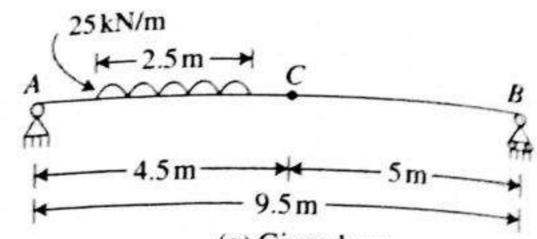
UDL LONGER THAN THE SPAN:

1. A UDL of intensity 15kN/m rolls across a girder of span 7m.the UDL occupies the entire span. Calculate the max. ’-‘ve and ‘+’ve S.F. and max B.M at a section 4m from the left support.

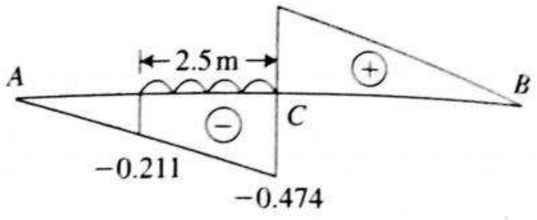
   Given a=4m;b=3m;l=7m;W=15kN/m
   We can obtain max .’-‘ve S.F when the load occupies the distance from left support and the section.
   max. ’-‘ve S.F=-17.14kN
   when the load is on the segment between the section and right support we get max.’+’ve S.F=9.64kN
   we get max. B.M when UDL occupies the entire span
   M_{MAX}=90KN-m

UDL SHORTER THAN THE SPAN:

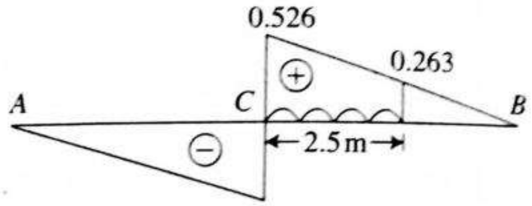
1.A UDL of length 2.5m and intensity 25kN/m rolls across a girder of span 9.5m shown in figure. Calculate the maximum negative and positive S.F and maximum at a section 4.5m from the left support.



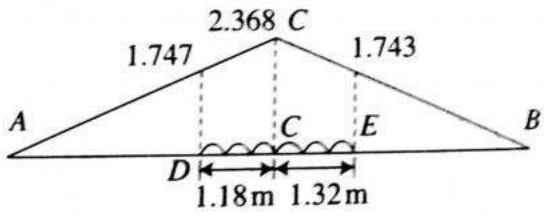
(a) Given beam



(b) Load position for −ve SF



(c) Load position for +ve SF



(d) Load position for BM

Let the section C be situated at a distance of 4.5m from the left support, i.e., a=4.5m

We place the head of the load section C shown in fig.b .

the ordinate of negative S.F at C is =-0.474.we can calculate the ordinate of negative S.F at the tail of the load from similar triangles as -0.211.the area of the trapezoidal influence diagram shown in fig.b. and the intensity of load would give the maximum negative S.F i.e.,

Max.'-'ve S.F =-21.31kN

To get the max.'+'ve S.F we place the tail of the load at the section C as in fig.c. the ordinate of '+'ve S.F at C is (5/9.5)=0.526. the ordinate at the head of load can be calculated from similar triangles as 0.263. the product of the trapezoidal area of the positive influence diagram shown in fig.c. and the intensity of load would yield the max.'+'ve S.F as max.’+’ve S.F=24.66kN

the maximum B.M is obtained by placing the load about C shown in fig.d.

M_{MAX}=128.55kN-m

Theory of Structures

III year-I Semester

Unit-III

Lateral Load Analysis using Approximation Methods

Learning Material

Introduction:

`When lateral loads are assumed to act from left to right, all joints undergo clockwise rotation. Consequently end rotations of the members are also clockwise except at fixed bases.

The sides sway due to lateral loads increases progressively from bottom towards the top of the frame.

From the above configuration, the end moments in columns are anticlockwise and those in all beams are in clockwise.

The axial forces in left exterior columns on windward side are tensile whereas in right exterior columns on the leeward side are compressive.

In symmetrical frames, the axial forces in the central columns lying on axis of symmetry are zero.

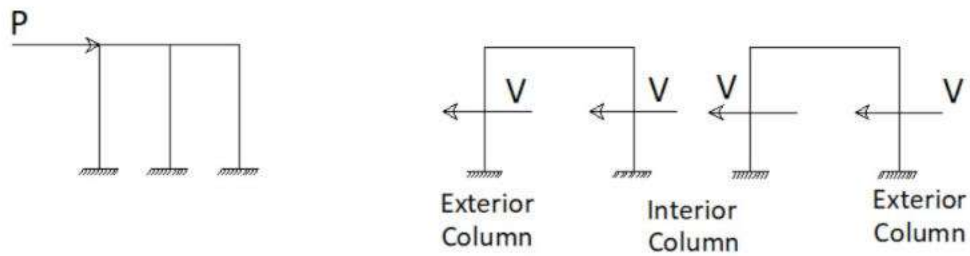
The axial forces in all beams are compressive. The vertical shear in each beam is uniform throughout

PORTAL METHOD

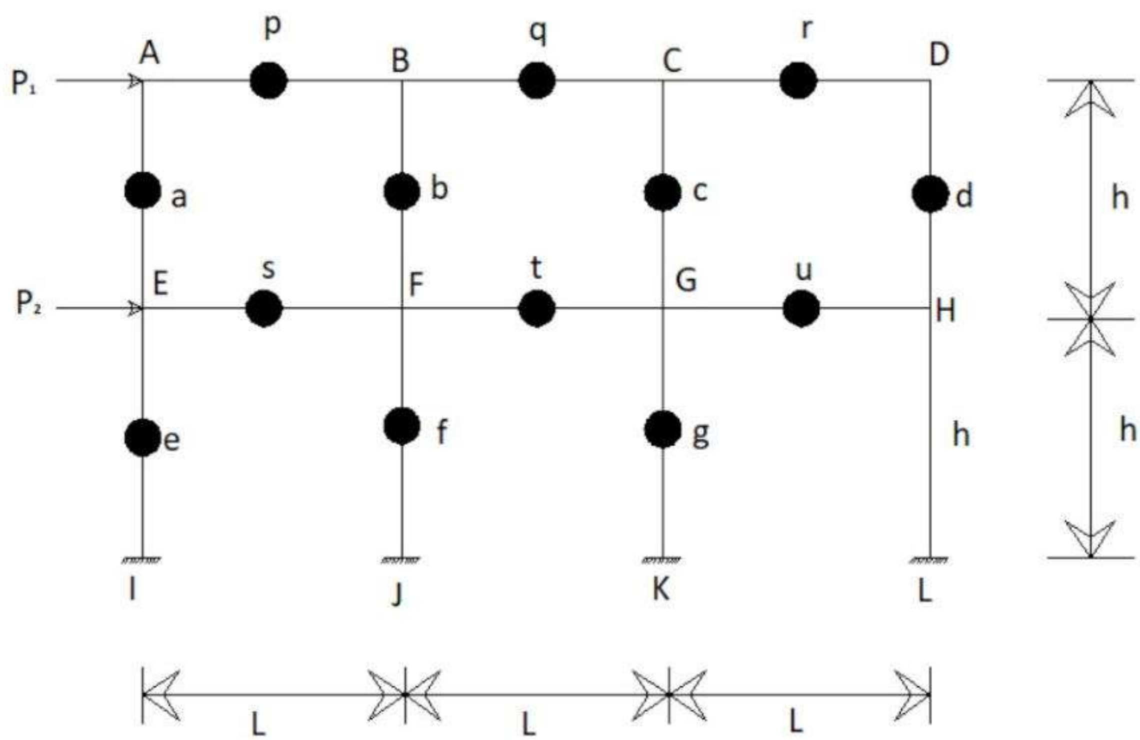
It is assumed that

1) Points of contra flexure occur at mid points of the beam members
2) Points of contraflexure occur at mid points of the columns. This assumption does not apply to columns with pinned bases where the moment is zero.

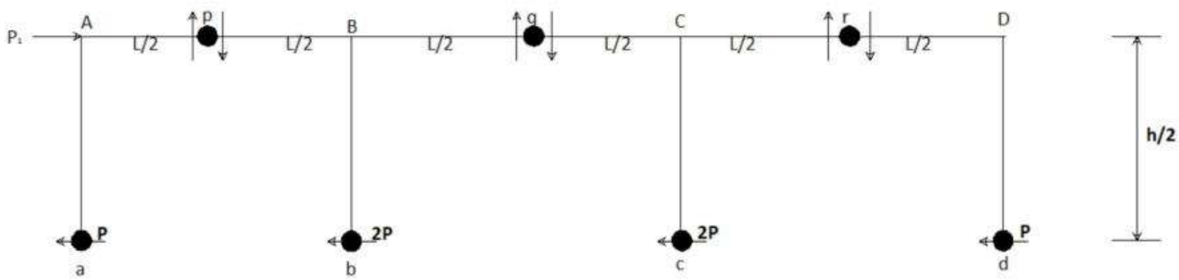
3) The horizontal shear taken by each interior column is double the horizontal shear taken by each exterior column.



Ex: Analyze the following frame by the portal method and draw final moment diagram.



Assume that inflexion points are at midpoint of all the members.
Let a plane is passed through abcd and the upper part is isolated as shown in figure.

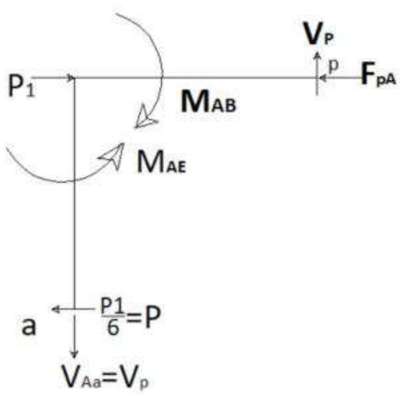


$P+2P+2P+P=P_1$

$$6P=P_1$$

$P=\frac{P_1}{6}$

In frame aAp



$F_{pA}=P_1-\frac{P_1}{6}=\frac{5P_1}{6}$

Taking moments about A

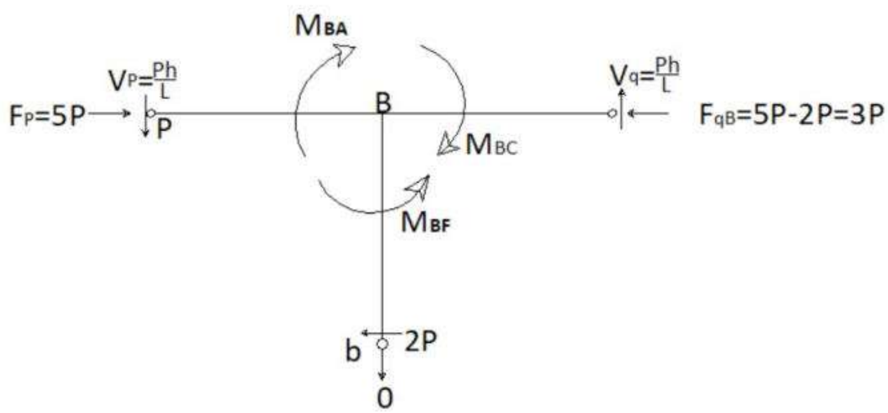
$\frac{P_1}{6}\times\frac{h}{2}=V_p\times\frac{L}{2}$

$$V_p=\frac{P_1}{6}\times\frac{h}{L}=\frac{Ph}{L}$$

$$M_{AE}=\frac{Ph}{2}$$

$$M_{AB}=\frac{ph}{L}\times\frac{L}{2}=\frac{Ph}{2}$$

Take joint B:



$$F_{qB}=5P-2P=3P$$

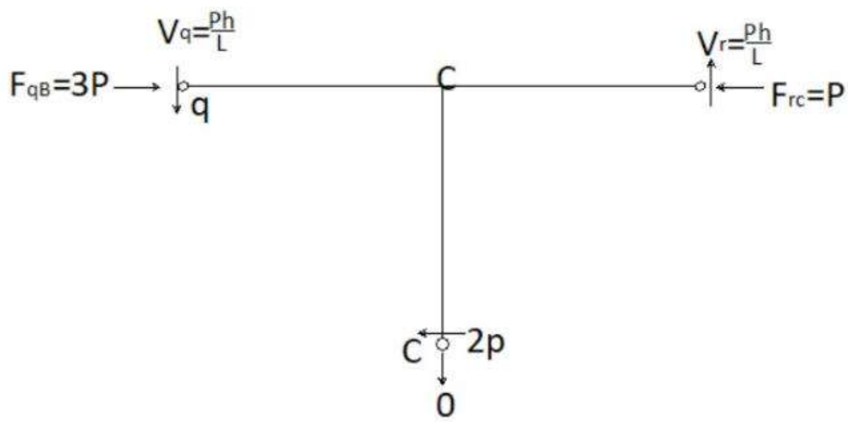
Taking moments about B

$$2P\times\frac{h}{2}=\frac{Ph}{L}\times\frac{L}{2}+V_q\times\frac{L}{2}$$

$$V_q\times\frac{L}{2}=Ph-\frac{Ph}{2}=\frac{Ph}{2}$$

$$V_q=\frac{Ph}{L}$$

Take Joint C:



$\sum H = 0$

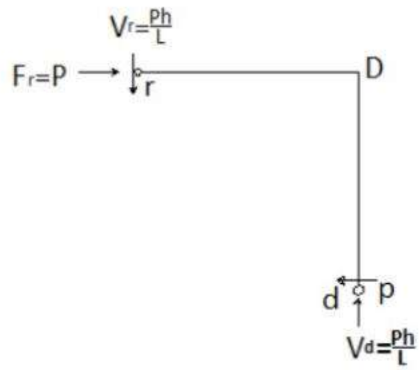
$F_{rC} = 3P - ¿2P = P$

Taking moments about C

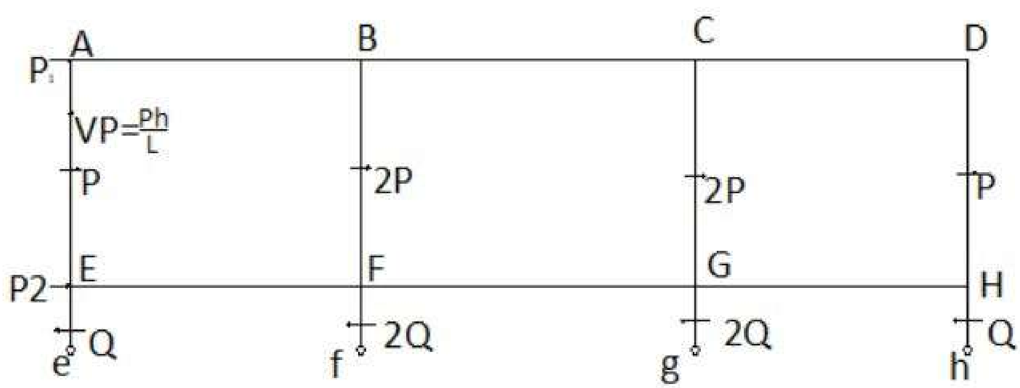
$$2P \times \frac{h}{2} = \frac{Ph}{L} \times \frac{L}{2} + V_q \times \frac{L}{2}$$

$$V_q = \frac{Ph}{2} \times \frac{2}{L} = \frac{Ph}{L}$$

Take joint D:



Consider plane passing through efgh:



6Q=6P+P2=P1+P2

$$Q=\frac{P1+P2}{6}$$

CANTILEVER METHOD

Assumptions:

1) Points of contra flexure in each member of beams and columns lies at its mid span or mid height

2) The axial stresses in the columns due to horizontal forces are directly proportional to their distance from the centroidal vertical axis of the frame.

PROCEDURE FOR CANTILEVER METHOD:

1) Consider the building frame shown above to the horizontal forces P_1 and P_2. Consider the top of storey free body diagram up to points of contra flexure of top storey columns.

2) Let H_1, H_2, H_3 and H_4 are horizontal shears in top storey columns and V_1, V_2, V_3 and V_4 be the axial forces in top storey columns.

Let a_1, a_2, a_3 and a_4 be the areas of cross section of columns

3) From Static equilibrium $P_1 = H_1 + H_2 + H_3 + H_4$ ______________________________ (1)

4) From assumption (2)

$$\frac{\frac{V_1}{a_1}}{x_1}=\frac{\frac{V_2}{a_2}}{x_2}=\frac{\frac{V_3}{a_3}}{x_3}=\frac{\frac{V_4}{a_4}}{x_4} \text{ ____________________________________(2)}$$

Where x_1, x_2, x_3 and x_4 are the centroidal distances of columns from vertical centroidal axis of the frame.

5) Taking moments about the points of inter section of vertical centroidal axis and the top beam.

$$(H\text{¿¿}1+H_2+H_3+H_4)\frac{h}{2}=V_1x_1+V_2x_2+V_3x_3+V_4x_4\text{¿}$$

But $H_1+H_2+H_3+H_4=P$

$$V_1x_1+V_2x_2+V_3x_3+V_4x_4=\frac{Ph}{2} \qquad (3)$$

From equation (2) and (3), axial forces V_1, V_2, V_3 and V_4 can be determined.

6) In order to determine the values of H, take moments about the contra flexure $M_1 \in beam\ AB$

$$H_1\times\frac{h}{2}=V_1\times\frac{L}{2}$$

$$H_1=V_1\times\frac{L}{h}$$

Similarly taking moments about $M_2 \in beam\ BC$

$$(H ¿¿ 1+H_2)\frac{h}{2}=V_1\left(L_1+\frac{L_2}{2}\right)+V_2\times\frac{L_2}{2} ¿$$

$$H_1+H_2=\frac{\left[V_1L_1+(V_1+V_2)\frac{L_2}{2}\right]2}{h}$$

Similarly taking moments about M_2 and M_3 of beam BC and CD towards right.

H_3 and H_4 can be determined.

Ex:

Analyze the frame shown in fig by cantilever method. Take Cross section areas of all columns as the same

1) To calculate vertical centroidal axis of the frame AE

$$\acute{x}=\frac{(0\times a)+4a+9a+15a}{a+a+a+a}=\frac{28a}{4a}=7m$$

$$x_1=7m;\ x_2=3m;\ x_3=2m;\ x_4=8m$$

Consider top storey:

Taking moments about x (Intersection point of vertical centroidal axis and top beam)

$$(H¿¿1+H_2+H_3+H_4)\frac{h_1}{2}=V_1x_1+V_2x_2+V_3x_3+V_4x_4¿$$

$$7V_1+3V_2+2V_3+8V_4=12\times1.5=18¿(1)$$

Also $\frac{\frac{V_1}{a_1}}{x_1}=\frac{\frac{V_2}{a_2}}{x_2}=\frac{\frac{V_3}{a_3}}{x_3}=\frac{\frac{V_4}{a_4}}{x_4}$

$$\frac{V_1}{7}=\frac{V_2}{3}=\frac{V_3}{2}=\frac{V_4}{8}$$

$$V_2=\frac{3}{7}\times V_1;V_3=\frac{2}{7}\times V_1;V_4=\frac{8}{7}\times V_1$$

Substituting in equation (1)

$$7V_1+3\times\frac{3}{7}\times V_1+2\times\frac{2}{7}\times V_1+8\times\frac{8}{7}\times V_1=18$$

$$V_1\left(\frac{49+9+4+64}{7}\right)=18$$

$$V_1=1\,kN$$

$$V_2=\frac{3}{7}\times V_1=\frac{3}{7}\times1=0.429\,kN$$

$$V_3=\frac{2}{7}\times V_1=\frac{2}{7}\times1=0.286\,kN$$

$$V_4=\frac{8}{7}\times V_1=V_4=\frac{8}{7}\times1=1.143\,kN$$

Taking Moments about O_1

$$1.5\times H_1=2\times V_1$$

$$H_1=\frac{2}{1.5}=1.33\,kN$$

Taking moments about O_2

$(H¿¿1+H_2)1.5=(6.5\times 1)+(0.429\times 2.5)¿$

$H_1+H_2=\frac{7.5725}{1.5}=5.05$

$H_2=5.05-1.33=3.72\,kN$

Taking moments about O_4

$$1.5\times H_4=2\times V_4$$

$$H_4=\frac{3\times 1.143}{1.5}=2.29\,kN$$

Taking moments about O_3

$(H¿¿3+H_4)1.5=(8.5\times 1.143)+(2.5\times 2.86)¿$

$H_3+H_4=\frac{9.72+0.71}{1.5}=6.95$

$H_3=6.95-2.29=4.66\,kN$

$Check$

$H_1+H_2+H_3+H_4=P_1$

$1.33+3.72+2.29+4.66=12$

$Hence\,OK$

$M_{ae}=1.33\times 1.5=2$

$M_{ae}=2$

$V_{ab}=\frac{2}{2}=1$

$M_{bf}=3.72\times 1.5=5.58\,kN-m$

$M_{bc}=5.58-2=3.58\,kN-m$

$V_{bc}=V_1+V_2=1+0.429=1.429\,kN$

$M_{dh}=2.29\times 1.5=3.43\,kN-m$

$$M_{dc}=3.43\,kN-m$$

$$V_{cd}=\frac{3.43}{3}=1.143\,kN$$

$$M_{cg}=4.66\times1.5=5.58\,kN-m$$

$$M_{cd}=3.43\,kN-m$$

$$M_{cb}=6.99-3.43=3.56\,kN-m$$

$$V_{bc}=\frac{3.56}{2.5}=1.424\,kN$$

$$V_1^I x_1+V_2^I x_2+V_3^I x_3+V_4^I x_4=12(5)+24(2)$$

$$7V_1^I+3V_2^I+2V_3^I+8V_4^I=108$$

$$V_2^I=\frac{3}{7}V_1^I;\ V_3^I=\frac{2}{7}V_1^I;\ V_4^I=\frac{2}{7}V_1^I$$

Substituting

$$\frac{V_1^I}{7}(49+9+4+64)=108$$

$$V_1^I=\frac{108\times7}{126}=6\,kN$$

$$V_2^I=\frac{3}{7}V_1^I=2.57\,kN$$

$$V_3^I=\frac{2}{7}V_1^I=1.71\,kN$$

$$V_4^I=\frac{2}{7}V_1^I=6.86\,kN$$

$$V_{ab}+V_{ef}=6$$

$V_{ef} = 6 - 1 = 5\,kN$

$M_{ea} = 2$

$M_{ef} = 5 \times 2 = 10$

$M_{ei} = 10 - 2 = 8\,kN - m$

$H_1^I = \frac{8}{2} = 4\,kN - m$

$$3 + V_{ef} + V_2^I = V_{bc}. + V_{fg}$$

$$1 + 5 + 2.57 = 1.43 + V_{fg}$$

$V_{fg} = 7.14\,kN$

$$M_{fb} = 5.58$$

$M_{fc} = 10$

$M_{fg} = 7.14 \times 2.5 = 17.85$

$M_{fj} = 17.85 + 10 - 5.58 = 22.2$

Theory of Structures

III year-I Semester

Unit-IV

Moment Distribution Method

Learning Material

Methods of Analysis of Indeterminate Structures

Indeterminate structures can be solved by two methods
A. Force or Flexibility or Compatibility method
B. Displacement or stiffness or Equilibrium method

A. Force or Flexibility Method

Flexibility is the deformation due to unit force
In this forces are taken as unknowns and equations are obtained considering computability
Examples:
1. Consistant deformation method
2. Method of least work
3. Theorem of three moments
4. Castigliano's second theorem $\Delta = \frac{\partial U}{\partial W}$

B. Displacement or Stiffness method

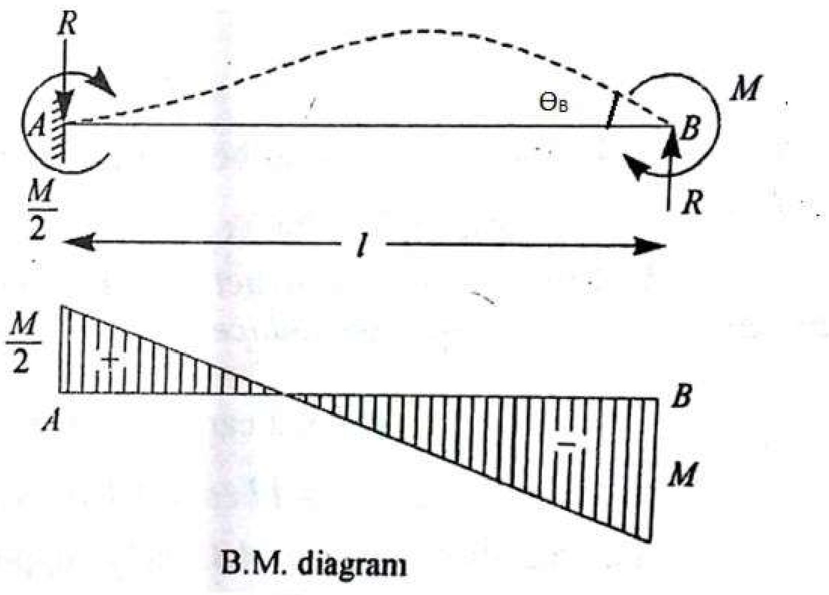
Stiffness is force for unit deformation
In this method, displacements are taken as unknowns and equations are obtained considering equilibrium of joints
Examples
1. Slope deflection method
2. Moment distribution method
3.Kani's method
4.Castigliano's first theorem $W = \frac{\partial U}{\partial \Delta}$

Moment Distribution Method

- Moment distribution method, also known as Hardy cross method, provides a convenient means of analyzing statically indeterminate beams and frames by simple hand calculations.
- This is basically an iterative process

- This is also a displacement method

1.Beam hinged at one end and fixed at other end :



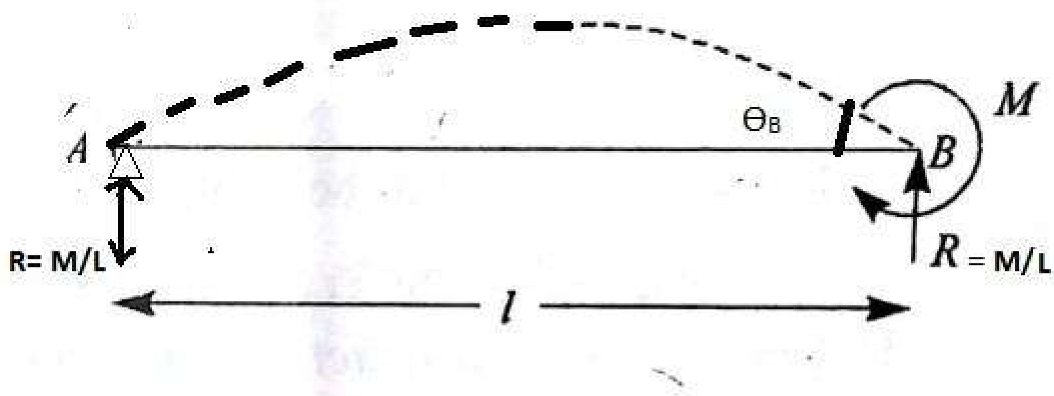
When moment M is applied at B ,the moment induced at fixed end at end A = M/2

Θ_B = ML/4EI

Stiffness = M/ Θ_B = 4EI/L

'k' stiffness is the moment required to rotate the near end of the beam through unit angle, without translation, the far end being fixed, is given by (4EI/L)

2.Beam supported freely at both ends



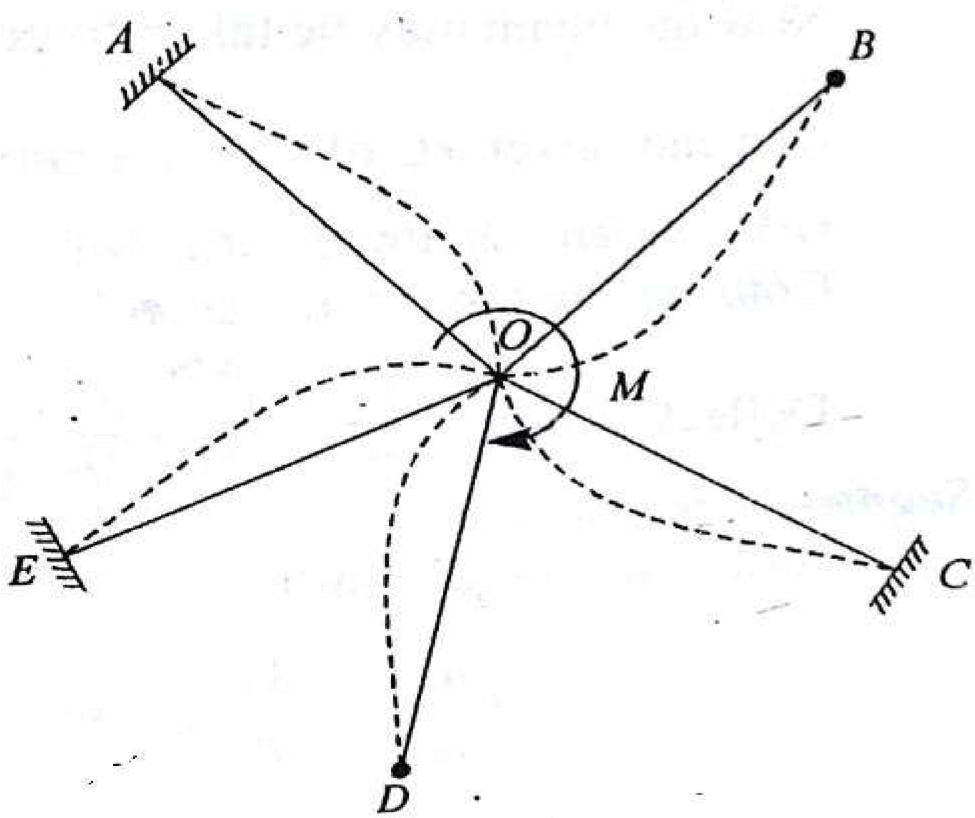
When a couple M is applied at B

Θ_B = ML/3EI

Stiffness of member = M/ Θ_B = 3EI/L

'k' stiffness is the moment required to rotate the near end of the beam through unit angle, without translation, the far end being freely supported is given by (3EI/L)

Distribution Theorem :



Let a moment M is applied to a structural joint to produce rotation without translation gets distributed among the connecting members at the joint in the same proportion as their stiffness.

Let OA,OB,OC,OD and OE be rigidly connected at joint O.

Due to moment M, the slope at O for each member is same.

Let far ends B and D are hinged while the others are fixed

Let l_1, l_2, l_3, l_4, l_5 – length of members

I_1, I_2, I_3, I_4, I_5 – moments of inertia

E – youngs Moduls same for all members

Let Θ_{OA} , Θ_{OB} , Θ_{OC} , Θ_{OD} , Θ_{OE} be the slopes of members at O

Total M = $M_1 + M_2 + M_3 + M_4 + M_5$

$\Theta_{OA} = M_1L_1/4EI_1$ (Far end fixed) ; $\Theta_{OB} = M_2L_2/3EI_2$ (far end hinged) ;

$\Theta_{OC} = M_3L_3/4EI_3$; $\Theta_{OD} = M_4L_4/3EI_4$ & $\Theta_{OE} = M_5L_5/4EI_5$

But $\Theta_{OA} = \Theta_{OB} = \Theta_{OC} = \Theta_{OD} = \Theta_{OE}$

$$\frac{M_1 l_1}{4E\ I_1} = \frac{M_2 l_2}{3E\ I_2} = \frac{M_3 l_3}{4E\ I_3} = \frac{M_4 l_4}{3E\ I_4} = \frac{M_5 l_5}{4E\ I_5}$$

$$\frac{M_1}{\left(\frac{4E\ I_1}{l_1}\right)} = \frac{M_2}{\left(\frac{3E\ I_2}{l_2}\right)} = \frac{M_3}{\left(\frac{4E\ I_3}{l_3}\right)} = \frac{M_4}{\left(\frac{3E\ I_4}{l_4}\right)} = \frac{M_5}{\left(\frac{4E\ I_5}{l_5}\right)}$$

$$M_1 : M_2 : M_3 : M_4 : M_5 = \frac{4E\ I_1}{l_1} : \frac{3E\ I_2}{l_2} : \frac{4E_3\ I_3}{l_3} : \frac{3E,\ I_4}{l_4} : \frac{4E\ I_5}{l_5}$$

Or $M_1/k_1 = M_2/k_2 = M_3/k_3 = M_4/k_4 = M_5/k_5$

Total stiffness at joint O , $k = k_1 + k_2 + k_3 + k_4 + k_5$

$M_1 = (k_1/\Sigma k) . M$, $M_2 = (k_2/\Sigma k).M$, $M_3 = (k_3/\Sigma k).M$, $M_4 = (k_4/\Sigma k).M$, $M_5 = (k_5/\Sigma k).M$

The ratio $k_1/\Sigma k$ is called distribution factor of member OA at O

Distribution Factor for a member at a joint is the ratio of stiffness of the member to the total stiffness of the member to the total stiffness of all the members meeting at the joint.

Relative stiffness

We know that

$$M_1 : M_2 : M_3 : M_4 : M_5 = \frac{4E\ I_1}{l_1} : \frac{3E\ I_2}{l_2} : \frac{4E_3\ I_3}{l_3} : \frac{3E,\ I_4}{l_4} : \frac{4E\ I_5}{l_5}$$

Dividing by '4E'

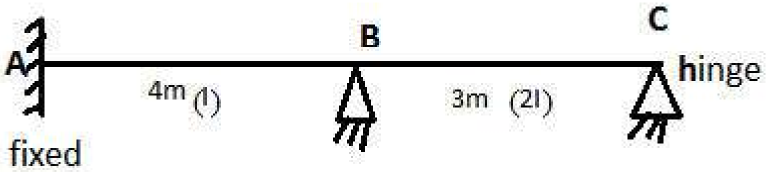
$M_1 : M_2 : M_3 : M_4 : M_4 : M_5 = I_1/ L_1 : 3I_2/ 4L_2): I_3/ L_3: 3I_4/4 L_4: I_5/ L_5$

Relative stiffness of the member at joint whose farther end is fixed is(I/L)

Relative stiffness of the member at joint whose farther end is hinge is(3 I/4L)

To Calculate distribution factors :

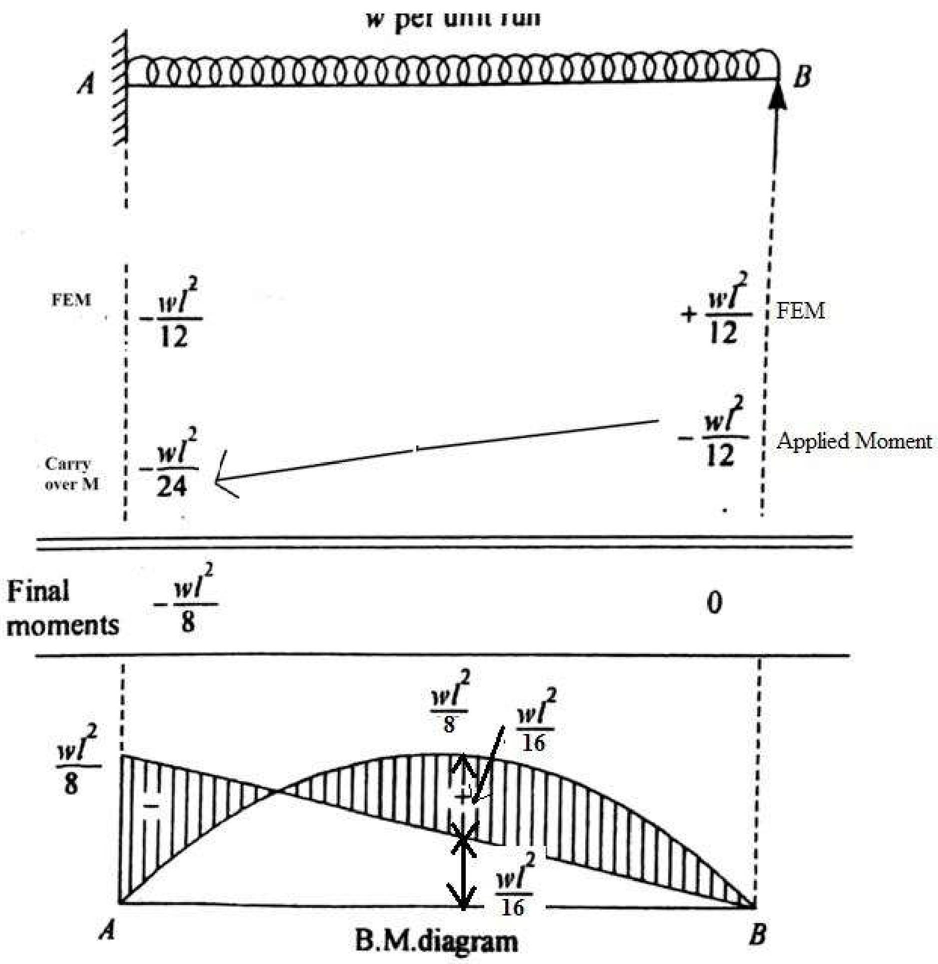
Example:



Joint	Member	Relative stiffness k	Total relative stiffness Σk	Distribution (D.F) factors (k/Σk)
B	BA	$\frac{I_{ab}}{l_{ab}} = \frac{I}{4}$	$\frac{3I}{4}$	$\frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{3}$
	BC	$\frac{3}{4}\frac{I_{bc}}{l_{bc}} = \frac{3}{4}\cdot\frac{2I}{3} = \frac{2I}{4}$		$\frac{\left(\frac{2}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{2}{3}$

Total:1.0

Problem : A beam AB of span L fixed at A and simply supported at B carries u.d.l of w per unit run over the entire span .Find the support moments and draw the B.M diagram



M_{AB} = - $wl^2/8$; M_{BA} = 0

Maximum positive moment = $wl^2/8$ - $wl^2/16$ = $wl^2/16$

Procedure :

1 Lock all the joints which are not fixed so that the span behaves like a fixed beam

2 Obtain fixed end moments(FEM)

3 Anticlockwise –ve , clockwise +ve (sign convertion)

4 Since the B.M at simply support at B is zero, apply a moment in opposite direction to make it zero

5 But ,since far end is fixed, the moment carried over to A = M/2 = - $wl^2/24$

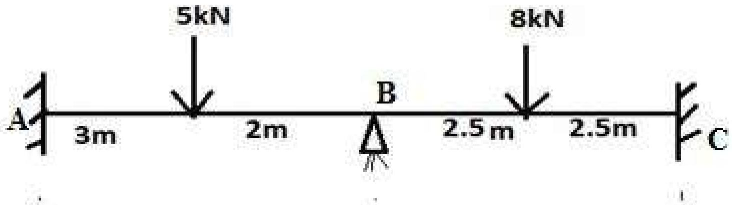
6 Obtain fixed moments at A = - $wl^2/12$ + - $wl^2/24$ = - $wl^2/8$

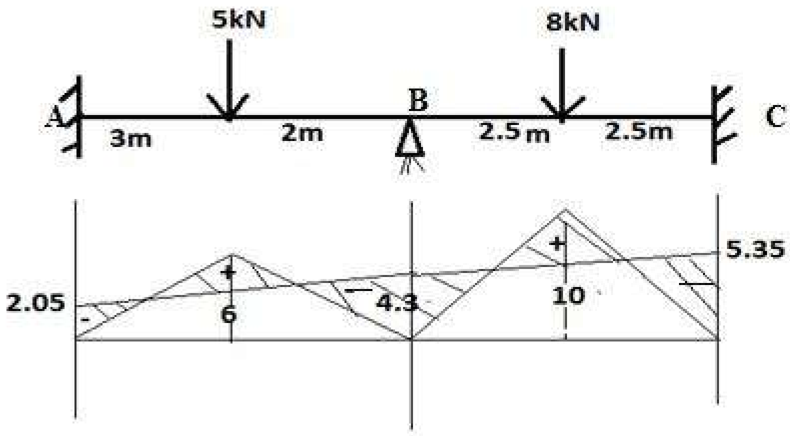
At B = $wl^2/12$- $wl^2/12$ = 0

Moment Distribution method

Problem

1.For the continuous beam, shown in figure , obtain the beam end moments and draw bending moment diagram .Flexural rigidity is same through out





JOINT AT B

	Moment	D.F		moment
		0.5	0.5	
FEM	-2.4	+3.6	-5.0	+5.0
Balance		+0.7	+0.7	
Carryover	+0.35	—	—	0.35
Final	-2.05	+4.3	-4.3	+5.35

Cycle of moment distribution :

1 calculate FEM

2 Balance the joint

3 Carryover the applied moment in proportion to their relative stiffness

This completes one cycle of moment distribution

Procedure

1 find the distribution factors for intermediate joints

2 lock all joints which are not fixed

3 Calculate FEM values

$\acute{M}_{AB}$ = -Wab^2/L^2 = 5x3x2x2/25 = -2.40

$\acute{M}_{BA}$ = Wa^{2}b/L^2 = 5x9x2/25 = +3.6

$\acute{M}_{BC}$ = - WL/8 = 8x5/8 = -5.0

$\acute{M}_{CB}$ = WL/8 = +5.0

4 Find the unbalance moments at B = +3.6-5.0 = -1.4 . This -1.4 kN-m keeps the joint B locked..Actually joint B is not fixed ans it is free to rotate.

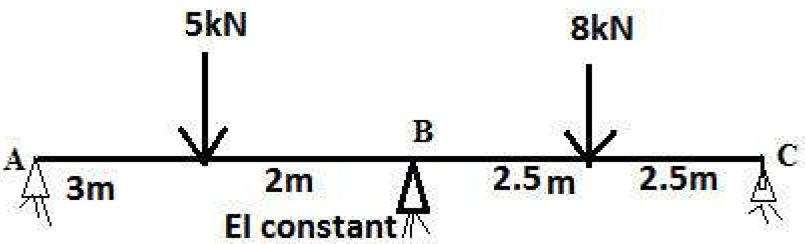
5 Therefore apply +1.4 kN-m to make it free .This balancing moment +1.4 kN-m should be distributed according to their distribution factors

6 Since joints A&C are fixed , no further applied moments and hence there is no need to carryover it to farther end

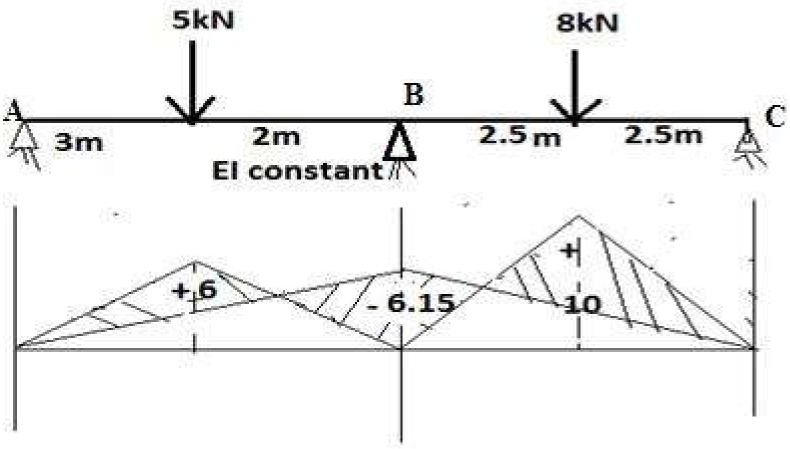
7 obtain the final moments

Problem .2

For previous example, when the ends are simply supported (hinged)



Joint	Member	Relative stiffness k	Total relative stiffness Σk	Distribution factors k/ Σk
B	BA	$\frac{I}{5}$	$\frac{2I}{5}$	1/2
	BC	$\frac{I}{5}$		1/2



Procedure:

1 Lock the joints which are not fixed

2 $\acute{M}_{AB}$ = -Wab²/L² = -2.40

$\acute{M}_{BA}$ = Wa²b/L² = +3.6

$\acute{M}_{BC}$ = - wl²/12 = -5.0

$\acute{M}_{CB}$ = wl²/12 = +5.0

Unbalance moment at joint B = -1.40

Balance the joints A,B&C and apply balance moment = +1.4

Distribute +0.7 & +0.7

Balance joint A , apply +2.40

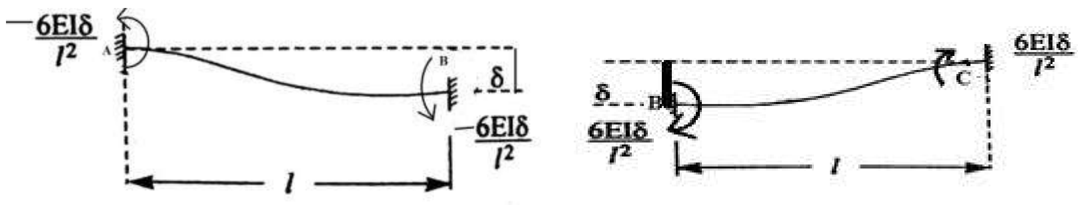
Balance joint C, apply -5.0

Stop the cycle when ΣM at A & B becomes zero

OR

At joint B				
cycle A	B		C	
	0.5	0.5		
-2.40	+3.60	-5.00	+5.0	**FEM**
+2.40			-5.0	**Balance at A&C**
	+1.20	-2.50		**carry over**
0	4.80	-7.50	0	
	+1.35	+1.35		**Balance**
0	+6.15	-6.15	0	**Final**

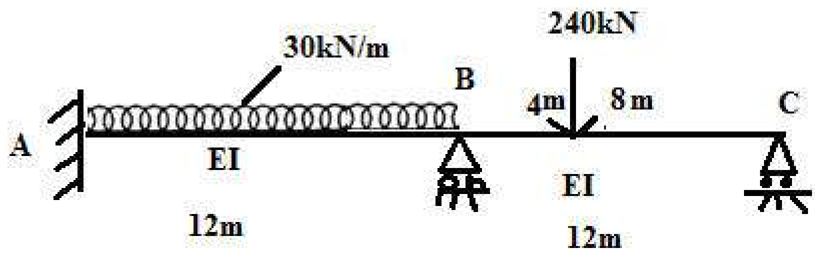
BEAM WITH SINKING OF SUPPORTS



In case of continuous beams, if any support sinks by δ then the final fixed end moment at each end will be the algebraic sum of the F.E.M carried by external loading and the settlement of supports.

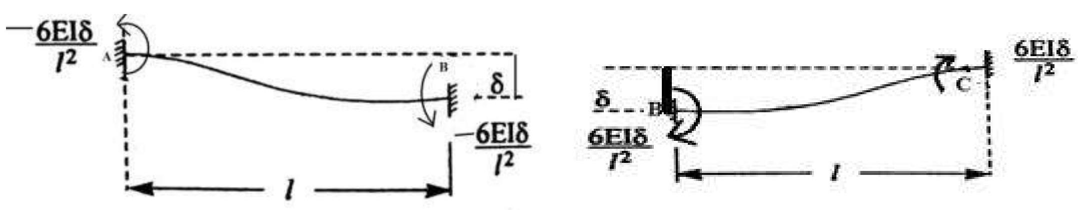
Problem 1

A continuous beam built-up at A and is carried over rollers at B&C as shown in figure. It carries u.d.l of 30KN/m over AB and point load of 240kN over BC,4m from the support B which sinks 30mm. Values of E and I are 200kN/mm^{2}and 2x10^9 mm4 respectively and uniform throughout .Calculate support moments and draw B.M and S.F diagrams.



Support B since by δ=30mm, E=200x10^8 kN/m^2

I=2x10^9 mm^4 or 2x10^{-3} m^4



Fixed End Moments :

$\acute{M}_{AB}$ = - $wL^2/12$- $6EI\delta/L^2$ = -30x144/12 - $6x2x10^8x2x10^{-3}x30x10^{-3}$ /144

= -360 -500 = -860kN-m

$\acute{M}_{BA}$ = +$wL^2/12$- $6EI\delta/L^2$ = 360 -500 = -140kN-m

$\acute{M}_{BC}$ = -Wab^2/L^2 + $6EI\delta/L^2$ = -240x4x64/144+ 500 = -426.7+500 = +73.3 kN-m

$\acute{M}_{CB}$ = Wa^2b/L^2+ $6EI\delta/L^2$ = +240x16x8/144 + 500 = +713.3 kN-m

Distribution Factors

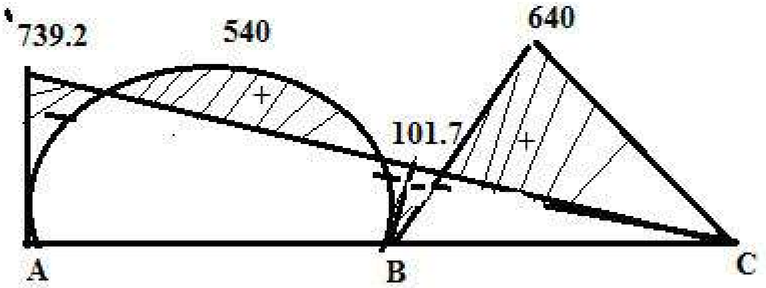
Joint	Member	Relative stiffness k	Total relative stiffness Σk	Distribution (D.F) factors (k/Σk)
B	BA	$\frac{I}{12}$	7I/48	0.571
	BC	$\frac{3}{4}\frac{I_1}{12} = \frac{I}{16}$		0.429
				Total = 1.0

Moment distribution Table

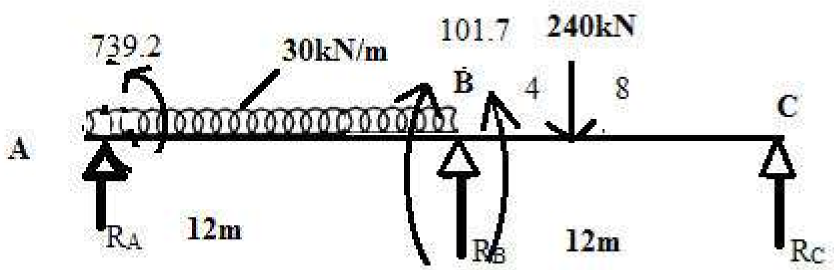
joint	A	B	B	C	
		0.571	0.429		
	-860.0	-140.0	+73.3	+713.3	FEM
				-713.3	Balance joint c
			-356.6		carry over
	-860.0	-140.0	-283.3	0	
		+241.7	+181.6		balance
	+120.8			0	carry over
	-739.2	+101.7	-101.7	0	Final moment

Free moment span AB = $wL^2/8 = 30x144/8 = +540$

Free moment span BC = $+Wab/L = 240x4x8/12 = +640$



B.M.D



Taking moments about B towards left

$12R_A + 101.7 = 739.20+(360x6)$

R_A =233.1kN

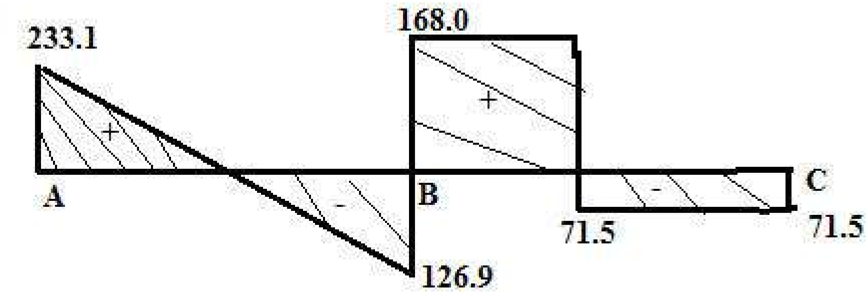
Taking moments about B towards right

$12\ R_C + 101.7 = 240 \times 4$

$R_C = 71.5$ kN

Total load = 360+240 = 600

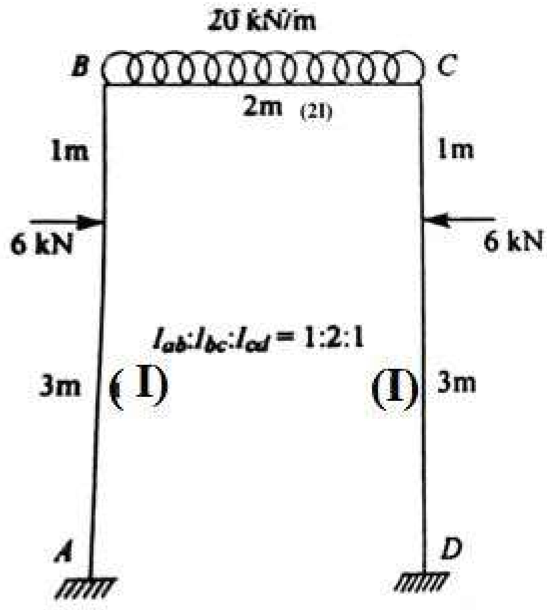
$R_B = 600 - 233.1 - 71.5 = 295.4$kN



S.F.D

Problem 2 on symmetrical frames

Analyse the portal frame shown in figure by moment distribution method and draw bending moment diagram.



M_{ab} = -6X3X1/16 = -1.125

M_{ba} = +6x9x1/16 = +3.375

M_{bc} = -20x4/12 = -6.67

M_{cb} = +6.67

M_{cd} = -3.375

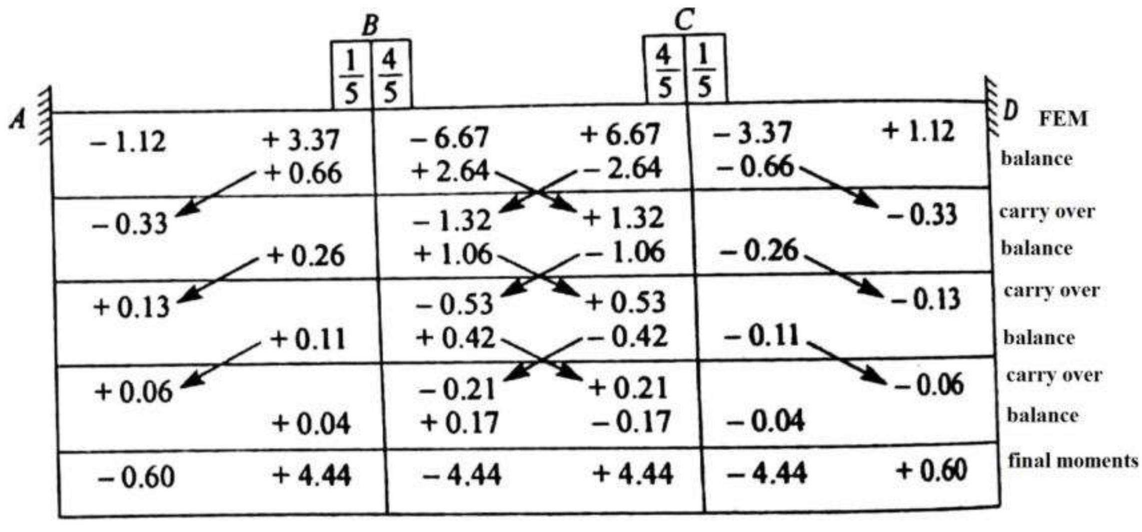
M_{dc}= +1.125

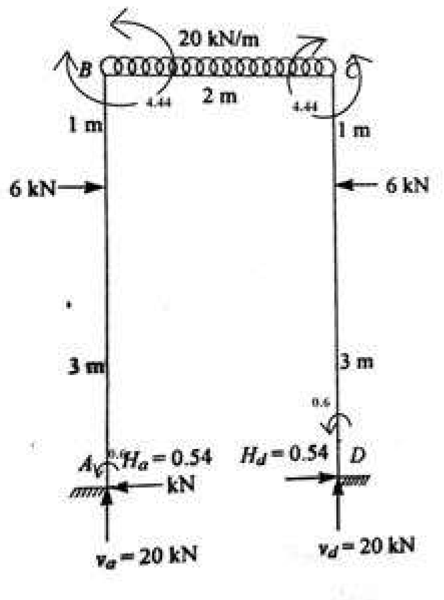
Distribution factors :

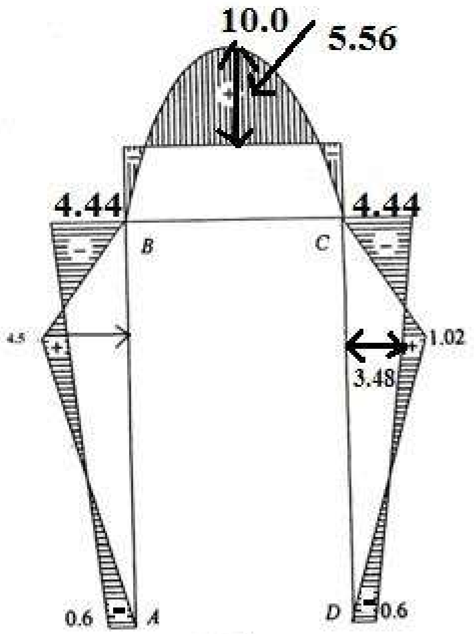
Distribution factors for joint B&C are same

Joint	*Member*	*Relative stiffness* k	*Total relative stiffness* Σk	*Distribution* (D.F) *factors* (k/Σk)
B	*BA*	$\frac{I}{4}$	5I/4	1/5
	BC	$\frac{2}{2}\frac{I_1}{} = I$		4/5
				Total = 1.0

Moment Distribution:







B.M. Diagram

Free moment in AB & CD = 6x3x1/4 = +4.5

In BC = 20x4/8 = +10

<u>At A &D</u>

Vertical reactions are = 2x20/2 = 20kN

For horizontal reaction at A,

Taking moments about B

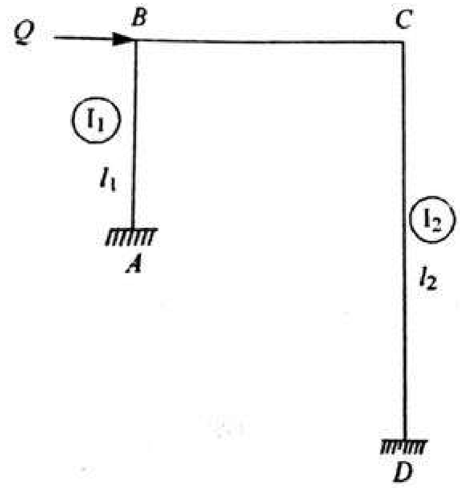
H_A =(-0.6 +4.4 -6x1)/4 = -2.16/4 = -0.54kN

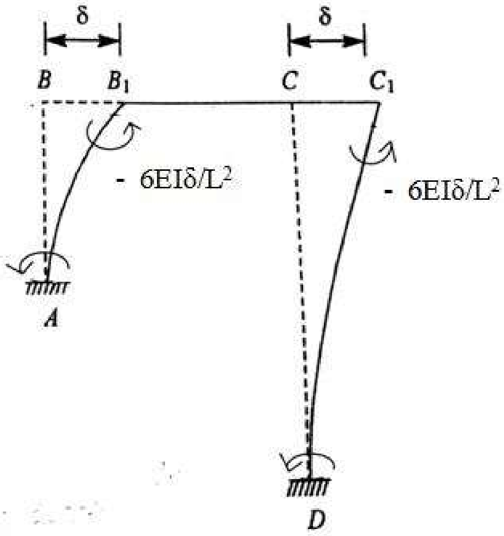
Similarly H_D = +0.54KN

Analysis of sway frames

Pure sway frame

<u>Case i) when both ends are fixed</u>





Frame subjected to pure sway force Q considering each member as fixed.

Member AB $\quad \overline{M}_{ab} = -\dfrac{6EI_1\delta}{l_1^2} = \overline{M}_{ba}$

Member CD $\quad \overline{M}_{cd} = -\dfrac{6EI_2\delta}{l_2^2} = \overline{M}_{dc}$

Since δ is not known, these moments are also not known.

But $\quad \overline{M}_{ba} : \overline{M}_{cd} = -\dfrac{6EI_1\delta}{l_1^2} : -\dfrac{6EI_2\delta}{l_2^2}$

$$= \frac{I_1}{l_1^2} : \frac{I_2}{l_2^2}$$

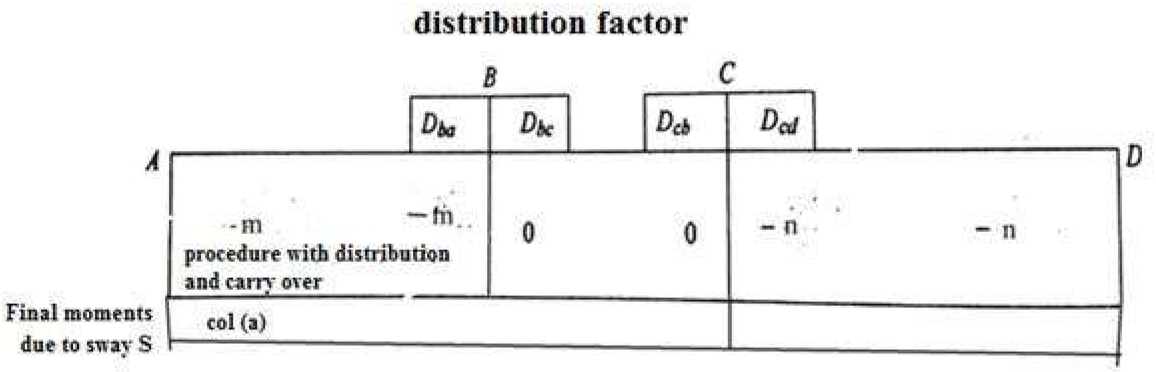
say $m : n$

Distribution Factors

A		B: D_{ba}	B: D_{bc}	C: D_{cb}	C: D_{cd}		D
	$-\dfrac{6EI_1\delta}{l_1^2}$	$-\dfrac{6EI_1\delta}{l_1^2}$	0	0	$-\dfrac{6EI_2\delta}{l_2^2}$	$-\dfrac{6EI_2\delta}{l_2^2}$	

Let $\quad \overline{M}_{ba} = -m$ and $\overline{M}_{cd} = -n$

$\therefore \quad \overline{M}_{ab} = -m$ and $\overline{M}_{dc} = -n$



Taking moments about B&C

H_A & H_D can be obtained

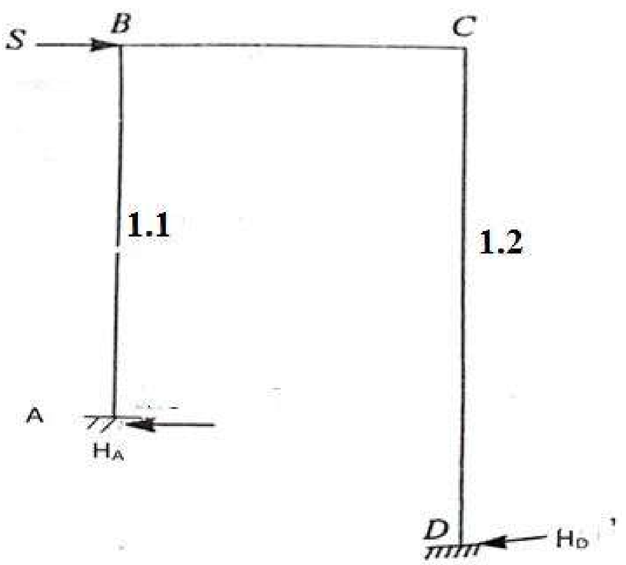
$S = H_A + H_D$

For a sway of S- moments col (a) are obtained

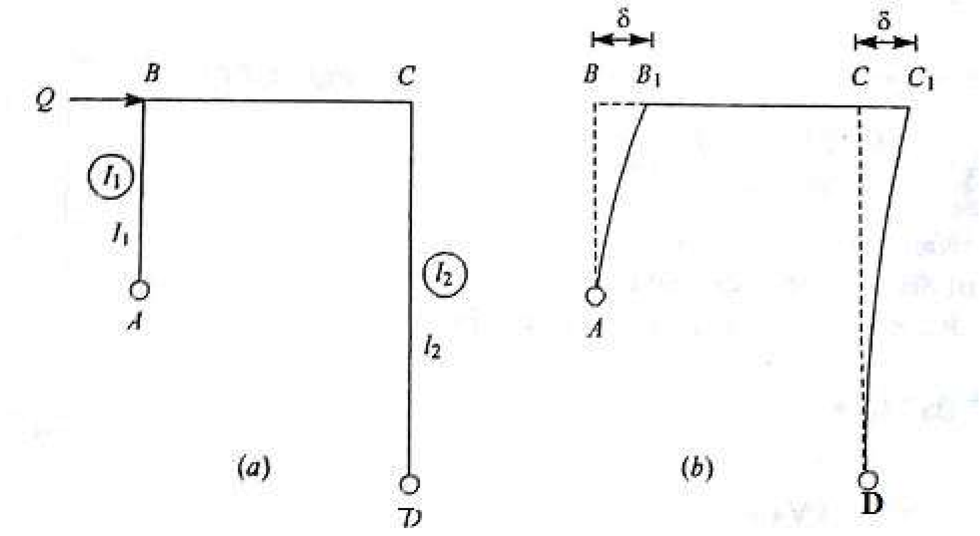
For actual sway force Q , the corresponding

final moments = (Q/S) col (a) Moments

Accordingly modify H_A & H_D also



Case 2) When both ends are hinged



$\acute{M}_{AB}$ = - 6EIδ/L^2 $\acute{M}_{BA}$ = - 6EIδ/L^2

$\acute{M}_{DC}$ = - 6EIδ/L^2

$\acute{M}_{CD}$ = - 6EIδ/L^2

A	B: D_{ba}	D_{bc}	C: D_{cb}	D_{cd}	D	
$-\frac{6EI_1\delta}{l_1^2}$	$-\frac{6EI_1\delta}{l_1^2}$	0	0	$-\frac{6EI_2\delta}{l_2^2}$	$-\frac{6EI_2\delta}{l_2^2}$	
$+\frac{6EI_1\delta}{l_1^2}$ →	$+\frac{3EI_1\delta}{l_1^2}$	0	0	$+\frac{3EI_2\delta}{l_2^2}$	← $+\frac{6EI_2\delta}{l_2^2}$	Apply Carry over
0	$-\frac{3EI_1\delta}{l_1^2}$	0	0	$-\frac{3EI_2\delta}{l_2^2}$	0	moments

These moments are entered in the moment table. The end A is corrected by applying a correcting moment of $+\frac{3EI_1\delta}{l_1^2}$ and the corresponding carry-over moment of $+\frac{3EI_1\delta}{l_1^2}$ is carried over to B. Similarly the end D is also corrected and a carry-over moment of $+\frac{3EI_2\delta}{l_2^2}$ is carried over to C. Adding the results, we get

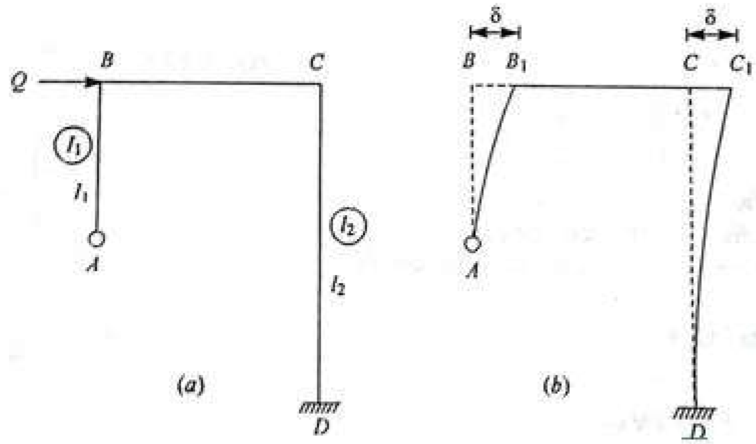
$$m'_{ab}=0\,;\; m'_{ba}=-\frac{3EI_1\delta}{l_1^2}\,;\; m'_{bc}=0=m'_{cb}\,,\quad m'_{cd}=-\frac{3EI_2\delta}{l_2^2}\,;\; m'_{dc}=0$$

$$m'_{ba}:m'_{cd}=-\frac{3EI_1\delta}{l_1^2}:-\frac{3EI_2\delta}{l_2^2}=\frac{I_1}{l_1^2}:\frac{I_2}{l_2^2}\text{ say } p:q$$

Further analysis is carried out as in case 1

case 3)When one end of the frame is hinged and other end is fixed

$$\overline{M}_{ab} = \overline{M}_{ba} = -\frac{6EI_1\delta}{l_1^2}\ ;\ \overline{M}_{cd} = \overline{M}_{dc} = -\frac{6EI_2\delta}{l_2^2}$$



	A	B: D_{ba}	B: D_{ba}	C: D_{cb}	C: D_{cd}	D
	$-\frac{6EI_1\delta}{l_1^2}$	$-\frac{6EI_1\delta}{l_1^2}$	0	0	$-\frac{6EI_2\delta}{l_2^2}$	$-\frac{6EI_2\delta}{l_2^2}$
Apply Carry Over	$+\frac{6EI_1\delta}{l_1^2}$	$+\frac{3EI_1\delta}{l_1^2}$	0	0		
Final	0	$-\frac{3EI_1\delta}{l_1^2}$	0	0	$-\frac{6EI_2\delta}{l_2^2}$	$-\frac{6EI_2\delta}{l_2^2}$

$$m'_{bc} = 0 = m'_{cb};\ m'_{cd} = -\frac{6EI_2\delta}{l_2^2} = m'_{dc}$$

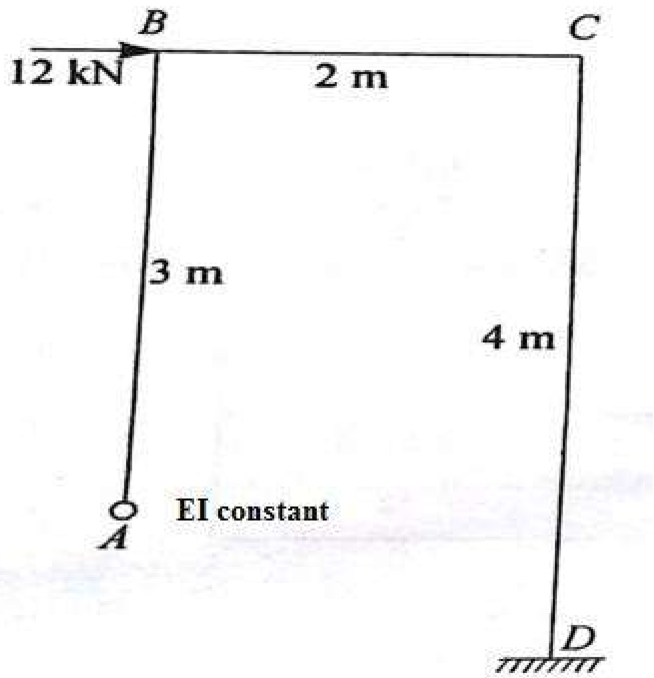
$$m'_{ba} : m'_{cd} = -\frac{3EI_1\delta}{l_1^2} : -\frac{6EI_2\delta}{l_2^2} = \frac{I_1}{l_1^2} : \frac{2I_2}{l_2^2}$$

$$say\ r : s$$

Problem 3:

Find the moments at A,B,C &D for the portal frame in figure .Draw also the B.M diagram for the frame .All the members of the frame have the same flexural rigidity

Distribution values:



Solution. **Distribution Factors.** These are calculated as shown in the table below :

Joint	*Member*	*Relative stiffness*	*Total relative stiffness*	*Distribution factors*
B	*BA*	$\frac{3}{4}\cdot\frac{I}{3}=\frac{I}{4}$	$\frac{3I}{4}$	$\frac{1}{3}$
	BC	$\frac{I}{2}=\frac{2I}{4}$		$\frac{2}{3}$
C	*CB*	$\frac{I}{2}=\frac{2I}{4}$	$\frac{3I}{4}$	$\frac{2}{3}$
	CD	$\frac{I}{4}$		$\frac{1}{3}$

$$m'_{ba} : m'_{cd} = -\frac{3EI_1\delta}{l_1^2} : -\frac{6EI_2\delta}{l_2^2} = \frac{I_1}{l_1^2} : \frac{2I_2}{l_2^2}$$

$m^{I}_{ba} : m^{I}_{cd} = I_1/(L_1)^2 : 2\,I_2/(L_2)^2 = I/3^2 : 2I/4^2 = I/9 : 2I/16$

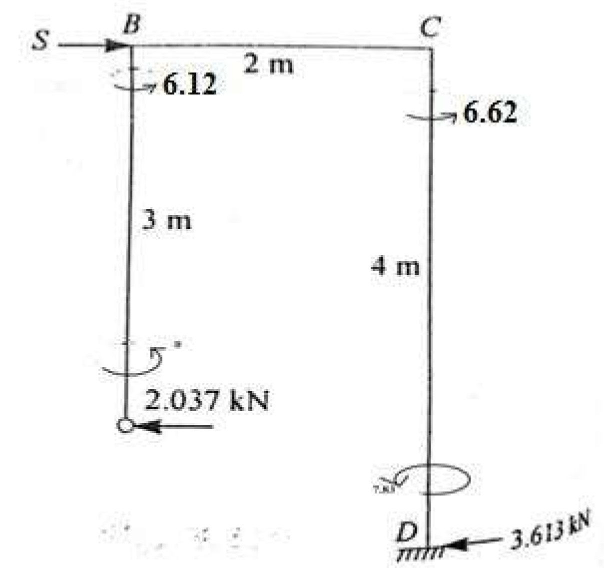
$m^{I}_{ba} : m^{I}_{cd} = 8:9$

Assume m^{I}_{ba} = -8.0kN-m m^{I}_{cd} = -9.0kN-m

Moments due to sway force S

Let S – Sway force for which sway moments are obtained

	A	B		C		D	
		1/3	2/3	2/3	1/3		
	0	−8.00	0	0	−9.00	−9.00	Assumed moment
		+2.67	+5.33	+6.00	+3.00		balance
			+3.00	+2.67		+1.50	carry over
		−1.00	−2.00	−1.78	−0.85		balance
			−0.69	−1.00		−0.45	carry over
		+0.30	+0.39	+0.67	+0.33		balance
			+0.33	+0.30		+0.17	carry over
		−0.11	−0.22	−0.20	−0.10		balance
			−0.10	−0.11		−0.05	carry over
		+0.03	+0.07	+0.07	+0.04		balance
Col (a)	0	−6.11	+6.11	+6.62	−6.62	−7.83	



Taking moments about B ,

$H_A x3 = 6.12$

$H_A = 6.12/3 = 2.04$ kN

Taking moments about C,

$H_D x4 = 6.62+7.83 = 14.45/4 = 3.61$ kN

Resolving horizontally

S = 2.04+3.61 = 5.65 kN ⇨

In table below , for a sway frame of 5.65 kN

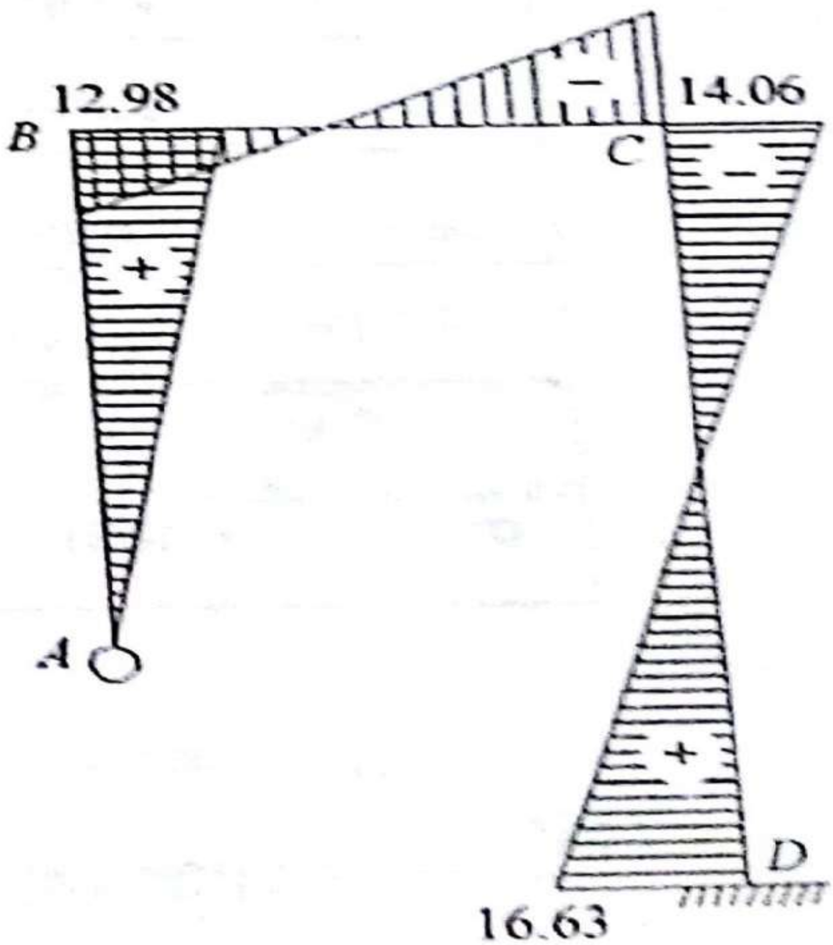
Sway moments are in col (a)

For actual sway of 12kN ,the corresponding sway moments are column (a) x(12/5.65)

	A		B		C	D
Col. (a)	0	−6.11	+6.11	+6.62	−6.62	−7.83
$\frac{12}{5.65}$ × col. (*a*) = Actual sway moments	0	−12.98	+12.98	+14.06	−14.06	−16.63

Corresponding $H_A = 12.98/3 = 4.33$ kN ⇦

Corresponding $H_D = 14.06 + 16.63/4 = 7.67$ kN ⇦

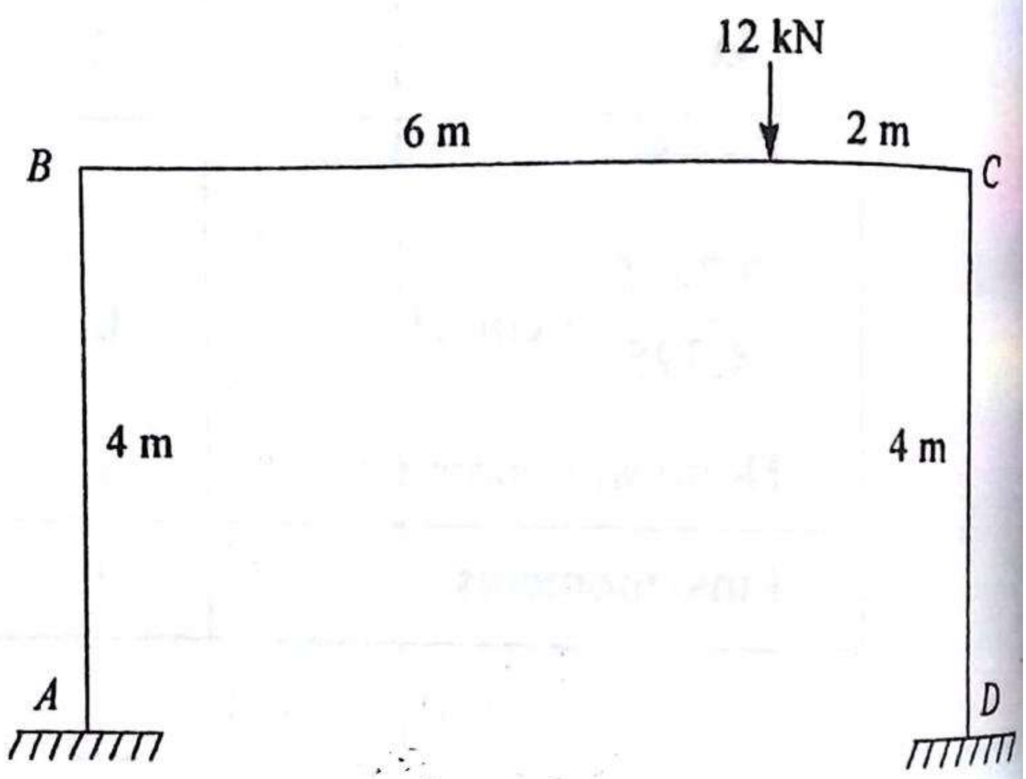


Analyse the portal frame shown in figure by moment distribution method . All members having same flexural rigidity.

$\acute{M}_{AB}$ =0 $\acute{M}_{BA}$¿0 $\acute{M}_{CD}$¿0 $\acute{M}_{DC}$ =0

$\acute{M}_{BC}$ = -12X4X16/64 = -4.5Kn-m

$\acute{M}_{CB}$ = 12X2X36/64 = +13.5kN-m



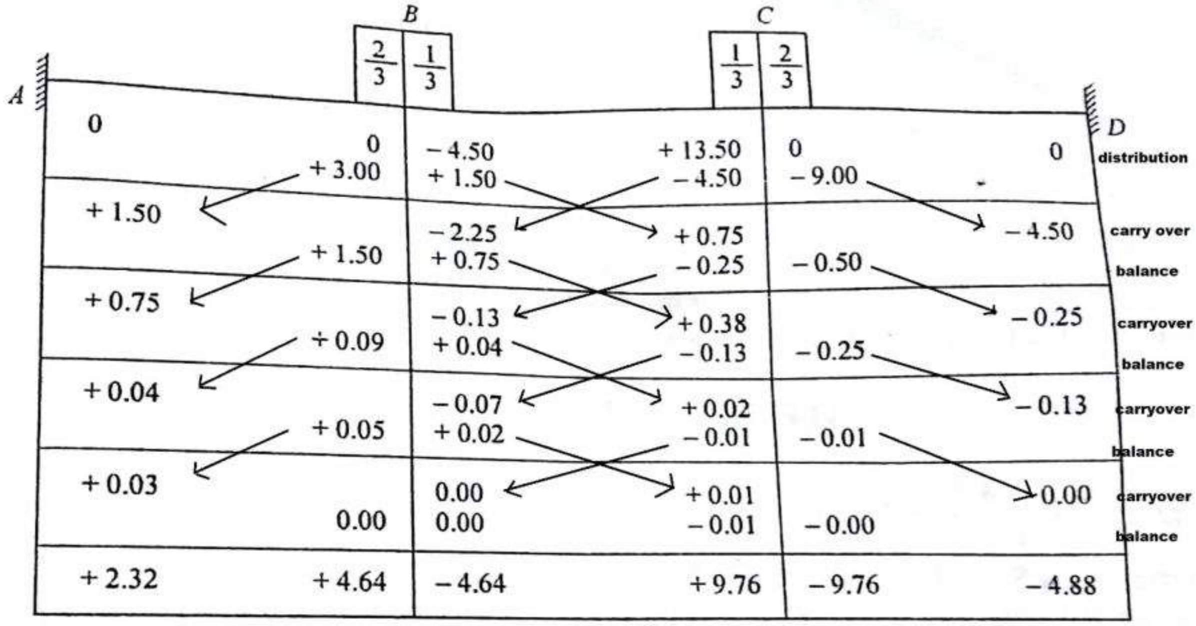
Distribution factors

Joint	Member	Relative stiffness	Total relative stiffness	Distribution factors
B	BA	$\frac{I}{4}=\frac{2I}{8}$	$\frac{3I}{8}$	$\frac{2}{3}$
	BC	$\frac{I}{8}$		$\frac{1}{3}$
C	CB	$\frac{I}{8}$		$\frac{1}{3}$
	CD	$\frac{I}{4}=\frac{2I}{8}$	$\frac{3I}{8}$	$\frac{2}{3}$

Non-Sway Analysis

Moment distribution for non -sway analysis

=

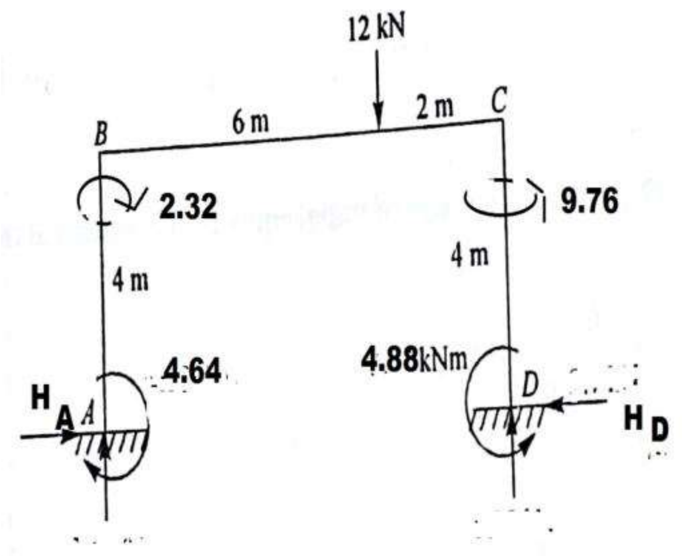


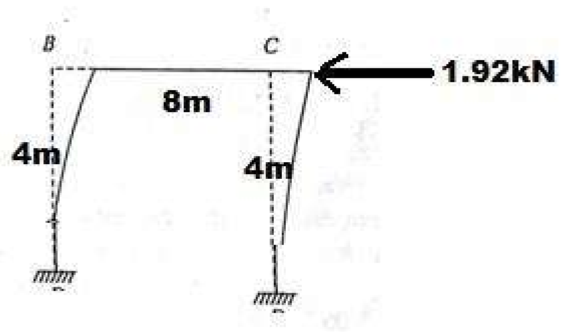
	A	B (2/3)	B (1/3)	C (1/3)	C (2/3)	D	
	0	0	−4.50	+13.50	0	0	
		+3.00	+1.50	−4.50	−9.00		distribution
	+1.50		−2.25	+0.75		−4.50	carry over
		+1.50	+0.75	−0.25	−0.50		balance
	+0.75		−0.13	+0.38		−0.25	carryover
		÷0.09	+0.04	−0.13	−0.25		balance
	+0.04		−0.07	+0.02		−0.13	carryover
		+0.05	+0.02	−0.01	−0.01		balance
	+0.03		0.00	+0.01		0.00	carryover
		0.00	0.00	−0.01	−0.00		balance
	+2.32	+4.64	−4.64	+9.76	−9.76	−4.88	

t

aking moments about B

H_A = (2.32+4.64)/4 = 1.74kN →





Taking moments about C

H_D = 9.76+4.88/4 = 3.66kN ←

Sway force = 3.66-1.74 = 1.92 kN ←

Sway analysis

$$M'_{ba} : M'_{cd} = \frac{I_1}{l^2} : \frac{I_2}{l_2^2} = \frac{I}{4^2} : \frac{I}{4^2} = 1:1$$

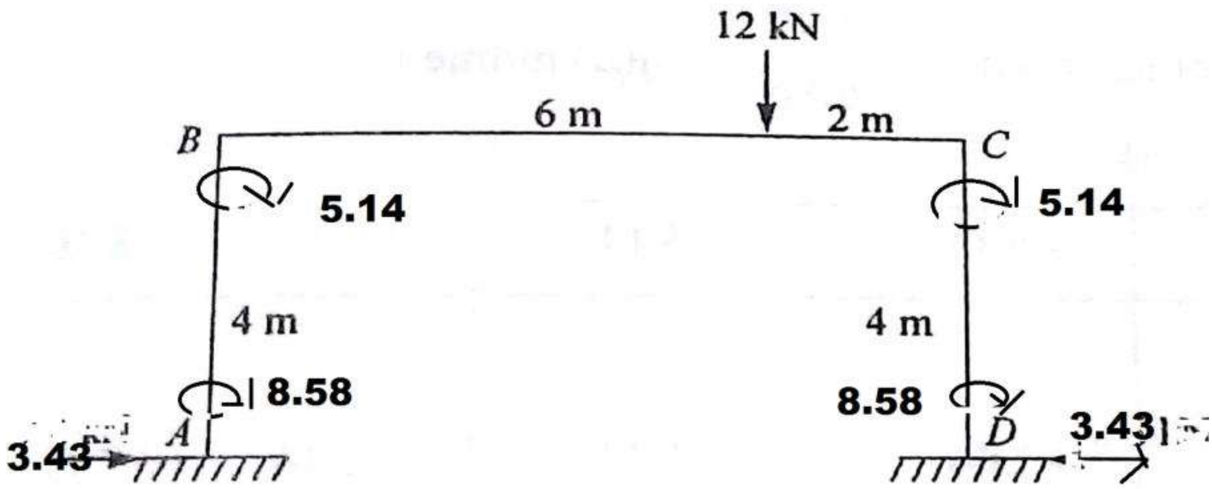
say 12 : 12

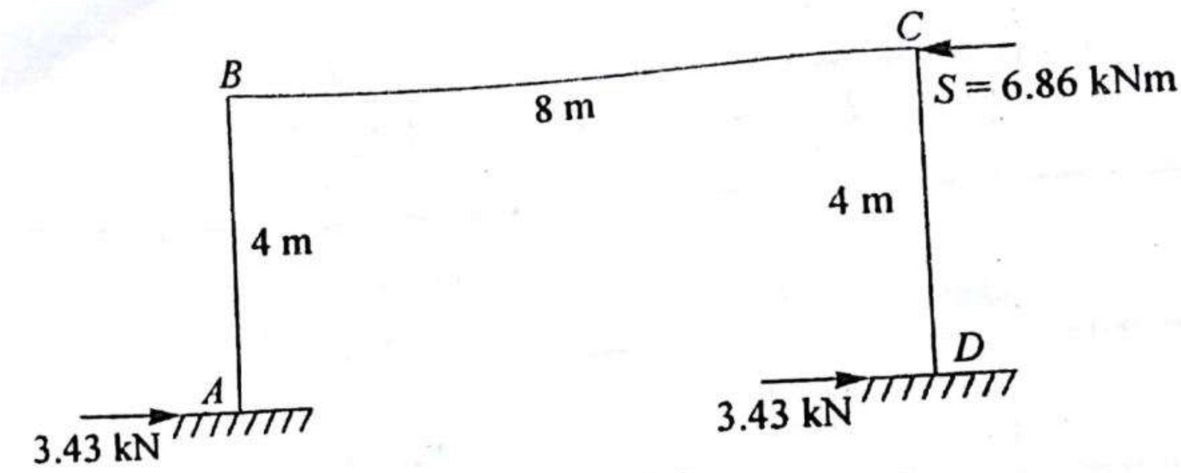
	A	B (2/3)	B (1/3)	C (1/3)	C (2/3)	D	
	+12.00	+12.00	0	0	+12.00	+12.00	Assumed FEM due to sway
		−8.00	−4.00	−4.00	−8.00		balance
	−4.00		−2.00	−2.00		−4.00	carry over
		+1.33	+0.67	+0.67	+1.33		balance
	+0.67		+0.33	+0.33		+0.67	carry over
		−0.22	− 0.11	+0.11	−0.22		balance
	−0.11		−0.06	−0.06		−0.11	carry over
		+0.04	+0.02	+0.02	+0.04		balance
	+0.02		+0.01	+0.01		+0.02	carryover
		−0.01	−0.00	−0.00	−0.01		balance
Col (a)	+8.58	+5.14	−5.14	−5.14	+5.14	+8.58	final

Taking moments about B &C

H_A = H_B=(5.14+8.58)/4 = 3.43kN

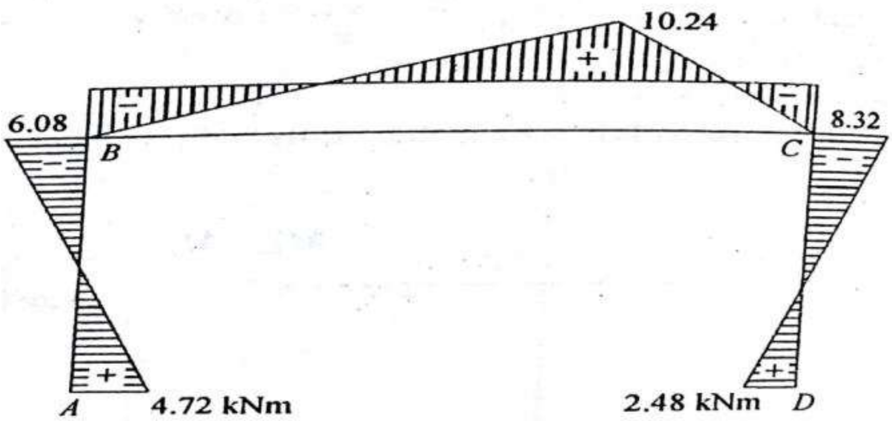
Sway S = 3.43+3.43=6.86Kn





For the actual sway force Q =1.92kN ,the actual sway moments obtained by (1.92/6.86) x col(a) moments

	A		B		C	
Col(a)	+8.58	+5.14	−5.14	+5.14	+5.14	+8.58
Actual sway moments $\frac{1.92}{6.86} \times$ col(a)	+2.40	+1.44	−1.44	−1.44	+1.44	+2.40
Non-sway moments	+2.32	+4.64	−4.64	+9.76	−9.76	−4.88
Final moments	+4.72	+6.08	−6.08	+ 8.32	−8.32	−2.48



B.M. Diagram

Theory of Structures

III year-I Semester

Unit-V

Kani's Method

Learning Material

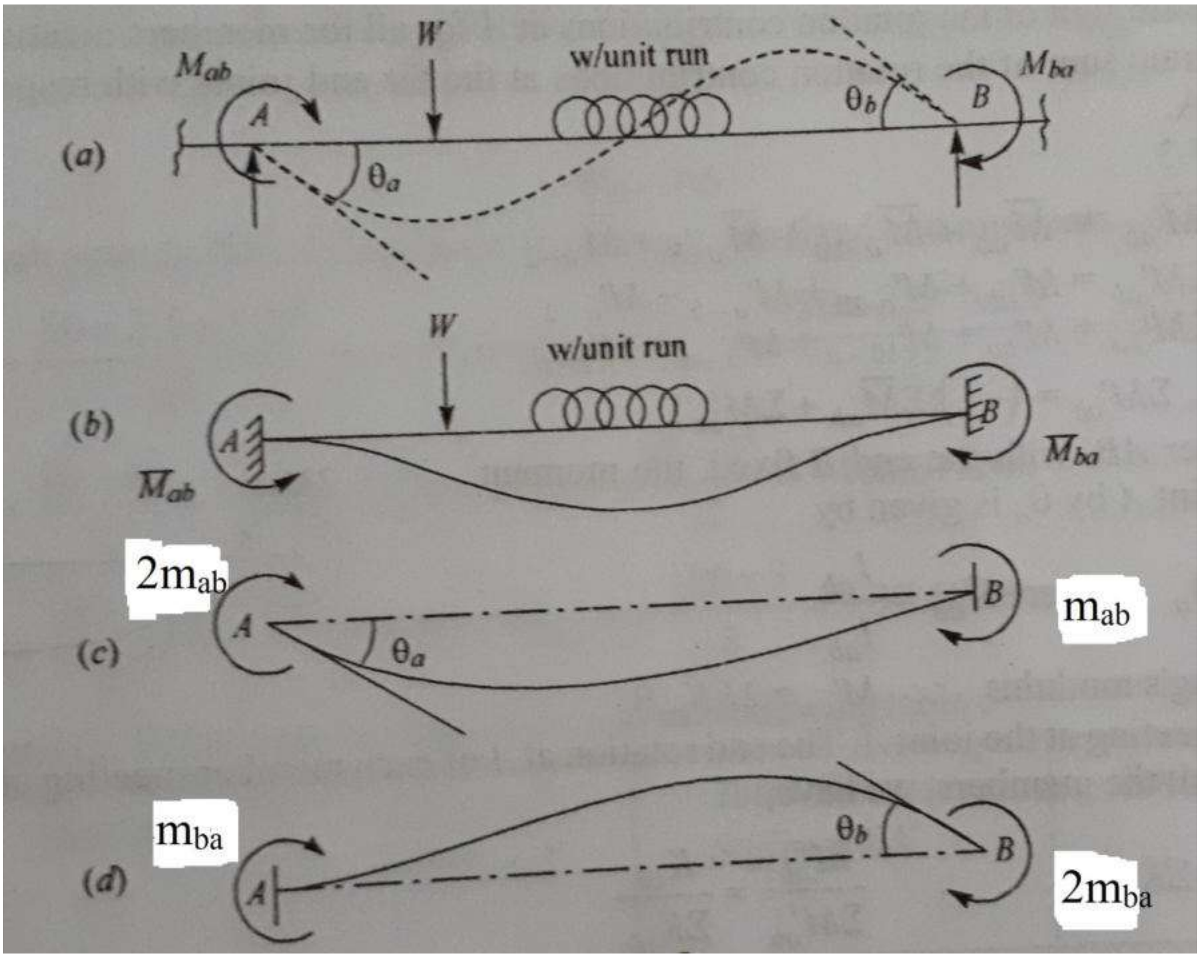
This method is developed by Gasper Kani of Germany. This method is displacement method.

Sign Convention:

1) Clockwise end moments are Positive
2) Clockwise rotations at ends are Positive

Let AB is a span of continuous beam and loaded.

Let M_{ab} and M_{ba} are the final moments.



Procedure:

1. Ends A and B are regarded as fixed and obtain fixed end moments $\acute{M}_{ab}$ and $\acute{M}_{ba}$.
2. Maintaining fixity at B, end A is rotated by θ_A by applying a moment $2m_{ab}$ at A, for this condition, a moment equal to m_{ab} is induced at far end B. This m_{ab} is the rotation contribution of end A.
3. Now, the end A is fixed and B is rotated by an angle θ_B applying a moment $2m_{ba}$ at B. For this condition, a moment of m_{ba} is induced at far end A. The moment m_{ba} is the rotation contribution of end B.

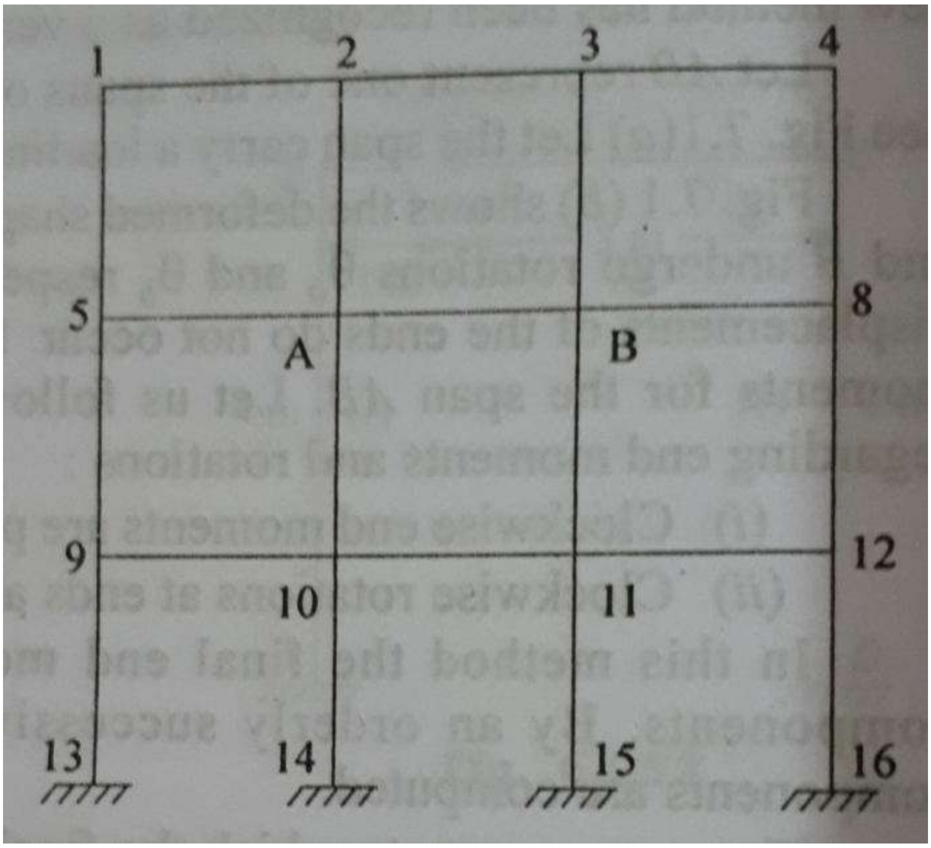
The Final moments are

$$M_{ab}=\acute{M}_{ab}+2m_{ab}+m_{ba}$$

$$M_{ba}=\acute{M}_{ba}+2m_{ba}+m_{ab}$$

———————— (i)

Now consider a multi storied frame



Now consider various members meeting at A.

End moments at A for members meeting at A are

$$M_{ab} = \acute{M}_{ab} + 2m_{ab} + m_{ba}$$

$$M_{a-10} = \acute{M}_{a-10} + 2m_{a-10} + m_{10-a}$$

$$M_{a-5} = \acute{M}_{a-5} + 2m_{a-5} + m_{5-a}$$

$$M_{a-2} = \acute{M}_{a-2} + 2m_{a-2} + m_{2-a}$$

_______________(ii)

For equilibrium of joint A, $\sum M_{ab} = 0$

$\sum M_{ab} = \sum \acute{M}_{ab} + 2\sum m_{ab} + \sum m_{ba} = ¿0¿$ _______________________(iii)

Where,

$\sum \acute{M}_{ab} = Algebric \sum of ¿ end\ moments\ at\ A\ for\ all\ members\ meeting\ at\ A ¿$

$\sum m_{ab} = Algebric \sum of\ rotation\ contributions\ at\ A\ for\ all\ members\ meeting\ at\ A$

$\sum m_{ba} = Algebric \sum of\ rotation\ contributio\ ns\ at\ far\ end\ joints\ wit\ h\ respect ¿ joint\ A ¿$

$\sum \acute{M}_{ab} = ¿ \acute{M}_{ab} + \acute{M}_{a-10} ¿ + \acute{M}_{a-5} + \acute{M}_{a-2}$

$\sum m_{ab} = ¿ m_{ab} ¿ + m_{a-10} + m_{a-5} + m_{a-2}$

$\sum m_{ba} = ¿ m_{ba} ¿ + m_{10-a} + m_{5-a} + m_{2-a}$

From Equation (iii)

$\sum m_{ab} = ¿ \left(\frac{-1}{2}\right)\left[\sum \acute{M}_{ab} + \sum m_{ba}\right] ¿$ ______________________(iv)

From diagram

$$2m_{ab} = \frac{4EI_{ab}}{L_{ab}}\theta_A = 4Ek_{ab}\theta_A$$

$$\frac{I_{ab}}{L_{ab}} = k_{ab}$$

$m_{ab}=2\,E\,k_{ab}\,\theta_A$

Consider the members meeting at A. Since rotation is same, θ_A *for all members* and assuming E is same for all members

$$\sum m_{ab} = 2\,E\,\theta_A \sum k_{ab}$$

$$\frac{m_{ab}}{\sum m_{ab}}=\frac{k_{ab}}{\sum k_{ab}}$$

$$¿\, m_{ab}=\frac{k_{ab}}{\sum k_{ab}}\sum m_{ab} \quad \text{———————————————(v)}$$

Applying (iv) in equation (v)

$$m_{ab}=\left(\frac{-1}{2}\frac{k_{ab}}{\sum k_{ab}}\right)\left[\sum \acute{M}_{ab}+\sum m_{ba}\right]$$

Ratio $\left(\frac{-1}{2}\frac{k_{ab}}{\sum k_{ab}}\right)$ *is called rotation factor for member AB at joint A*

Let $\mu_{ab}=\left(\frac{-1}{2}\frac{k_{ab}}{\sum k_{ab}}\right)$ *is rotation factor*

$$m_{ab}=\mu_{ab}\left[\sum \acute{M}_{ab}+\sum m_{ba}\right] \quad \text{———————————————(vi)}$$

By successive application of equation (vi) various joint rotation contributions can be determined. For approximation the rotation contribution of far end member meeting at joint A to be zero, the rotation contribution at A for member AB

$$m_{ab}=\mu_{ab}\left[\sum \acute{M}_{ab}+0\right]$$

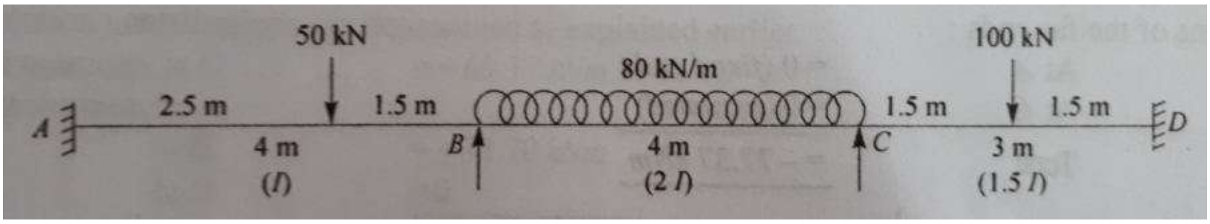
With approximate values of the rotation contributions computed, it is possible again to determine a more correct value of rotation contribution at A for the member AB using equation again $m_{ab}=\mu_{ab}\left[\sum \acute{M}_{ab}+\sum m_{ba}\right]$

This process can be continued till the present values are almost equal to previous values.

The final moments can be easily computed from the relation

$M_{ab} = \acute{M}_{ab} + 2m_{ab} + m_{ba}$

Ex: **Determine the support moments at A, B, C and D for a continuous beam shown in figure by Kani's method.**



$$\acute{M}_{ab} = \frac{-Wab^2}{l^2} = \frac{-50 \times 2.5 \times 1.5^2}{4^2} = -17.58\, kN-m$$

$$\acute{M}_{ba} = \frac{-Wa^2b}{l^2} = \frac{-50 \times 2.5^2 \times 1.5}{4^2} = 29.30\, kN-m$$

$$\acute{M}_{bc} = \frac{-wl^2}{12} = \frac{-80 \times 4^2}{12} - 106.67\, kN-m$$

$$\acute{M}_{cb} = \frac{wl^2}{12} = 106.67\, kN-m$$

$$\acute{M}_{cd} = \frac{-Wl}{8} = \frac{-100 \times 3}{8} = -37.5\, kN-m$$

$$\acute{M}_{dc} = \frac{Wl}{8} = \frac{100 \times 3}{8} = 37.5\, kN-m$$

Rotation factors at joint B and C:

Joint	Member	Relative Stiffness (k)	Total Stiffness $(\sum k)$	Rotation factor $\mu_{ab} = \left(\frac{-1}{2} \frac{k_{ab}}{\sum k_{ab}}\right)$
B	BA	I/4	3I/4	-1/6
	BC	2I/4		-1/3
C	CB	2I/4	I	-1/4
	CD	1.5I/3		-1/4

At joint B, $\sum \acute{M}_{ba} = \acute{M}_{ba} + \acute{M}_{bc} = 29.3 - 106.67 = -77.37\, kN-m$

At joint C, $\sum \acute{M}_{cb} = \acute{M}_{cb} + \acute{M}_{cd} = 106.67 - 37.5 = 69.17\, kN - m$

Initially the rotation contributions at A and D are Zero, since these ends are fixed.

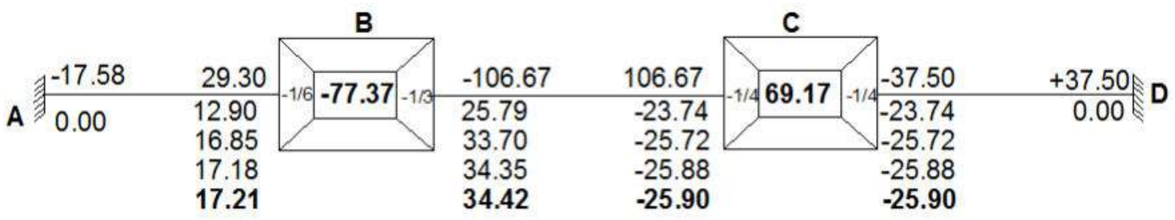
Rotation contribution values are determined by successive iteration process using equations

$$m_{ba} = \mu_{ba}\left[\sum \acute{M}_{ba} + \sum m_{ab}\right]$$

$$m_{bc} = \mu_{bc}\left[\sum \acute{M}_{bc} + \sum m_{cb}\right]$$

Assume $m_{ab} = m_{dc} = 0$

First trail:



Joint B,

$$m_{ba} = \frac{-1}{6}[-77.37] + 0 = 12.90$$

$$m_{bc} = \frac{-1}{3}[-77.37] + 0 = 25.79$$

Joint C,

$$m_{cb} = \mu_{cb}\left[\sum \acute{M}_{cb} + \sum m_{bc}\right]$$

$$m_{cb} = \frac{-1}{4}\left[(69.17) + 25.79 + 0\right] = \frac{-1}{4}[94.96] = -23.74$$

$$m_{cd} = \frac{-1}{4}[94.96] = -23.74$$

Second Trail:

Joint B,

$$m_{ba} = \frac{-1}{6}[-77.37 - 23.74] = 16.85$$

$$m_{bc}=\frac{-1}{3}[-77.37-23.74]=33.70$$

Joint C,

$$m_{cb}=\frac{-1}{4}[(69.17)+33.70]=-25.72$$

$$m_{cd}=\frac{-1}{4}[69.17+33.70]=-25.72$$

Third Trail:

Joint B,

$$m_{ba}=\frac{-1}{6}[-77.37-25.92]=17.18$$

$$m_{bc}=\frac{-1}{3}[-77.37-25.92]=34.35$$

Joint C,

$$m_{cb}=\frac{-1}{4}[(69.17)+34.35]=-25.88$$

$$m_{cd}=\frac{-1}{4}[69.17+34.35]=-25.88$$

Fourth Trail:

Joint B,

$$m_{ba}=\frac{-1}{6}[-77.37-25.88]=17.21$$

$$m_{bc}=\frac{-1}{3}[-77.37-25.88]=34.42$$

Joint C,

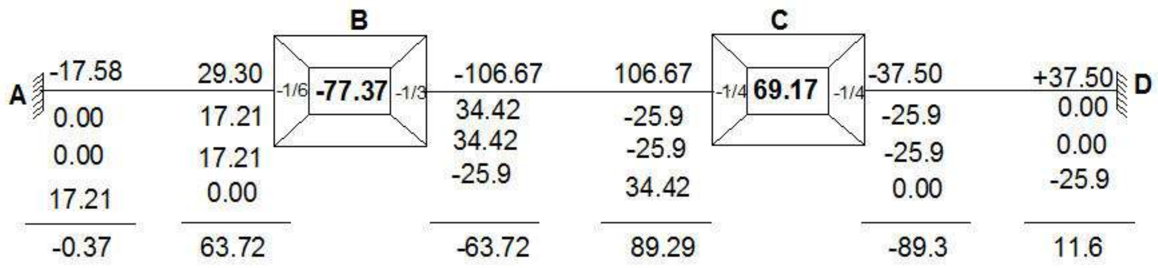
$$m_{cb}=\frac{-1}{4}[(69.17)+34.42]=-25.90$$

$$m_{cd}=\frac{-1}{4}[69.17+34.42]=-25.90$$

Values of 4^{th} trail and 3^{rd} trail are matching

$Final\ Moment\ M_{ab} = \acute{M}_{ab} + 2m_{ab} + m_{ba}$

¿¿ $End\ Moment + Twice\ t\ h\ e\ contribution\ of\ near\ end + Contribution\ of\ far\ end$



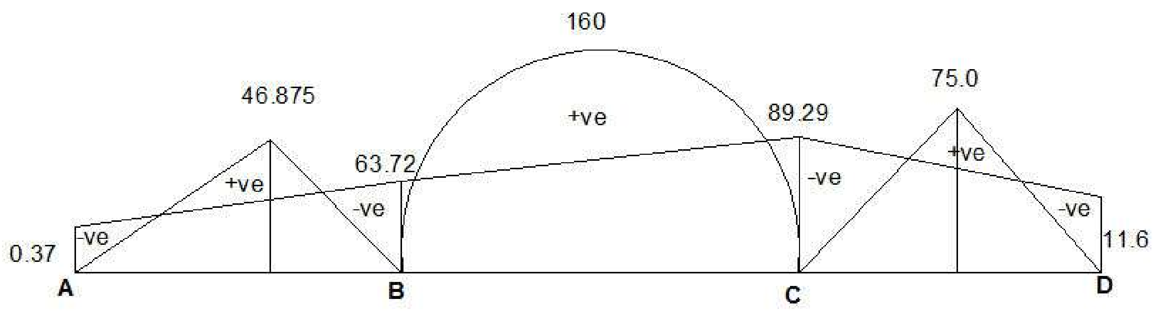
$Mid\ Span\ free\ Moments:$

¿ $span\ AB, free\ BM = \frac{Wab}{L} = \frac{50 \times 2.5 \times 1.5}{4} = 46.875\ kN - m$

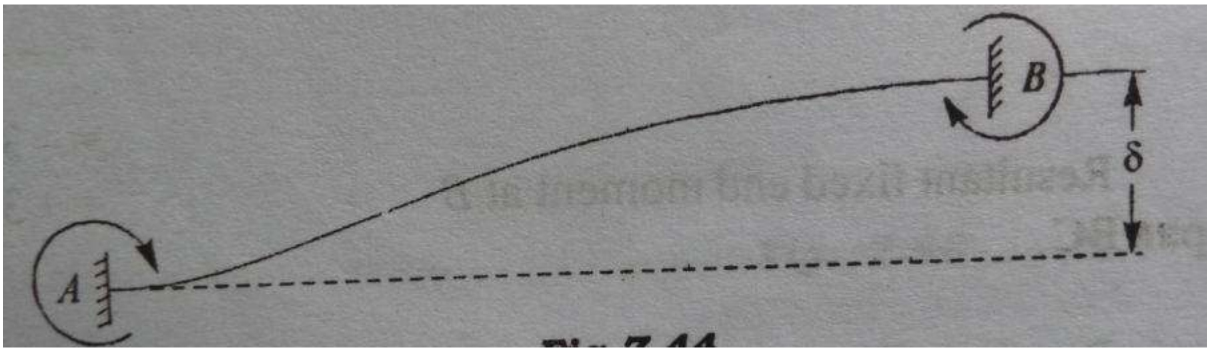
¿ $span\ BC, free\ BM = \frac{w\,l^2}{8} = \frac{80 \times 4^2}{8} = 160\ kN - m$

¿ $span\ CD, free\ BM = \frac{WL}{4} = \frac{100 \times 3}{4} = 75\ kN - m$

Bending Moment Diagram:



Members with relative lateral displacement:



Fixed end moments due to this condition (Sinking of supports)

$$m'_{ab} = m'_{ba} = \mp \frac{6EI\delta}{L^2}$$

When subjected to sinking of supports. The final moments at A and B are given by

$$M_{ab} = \acute{M}_{ab} + 2m_{ab} + m_{ba} + m'_{ab}$$

$$M_{ba} = \acute{M}_{ba} + 2m_{ba} + m_{ab} + m'_{ba}$$

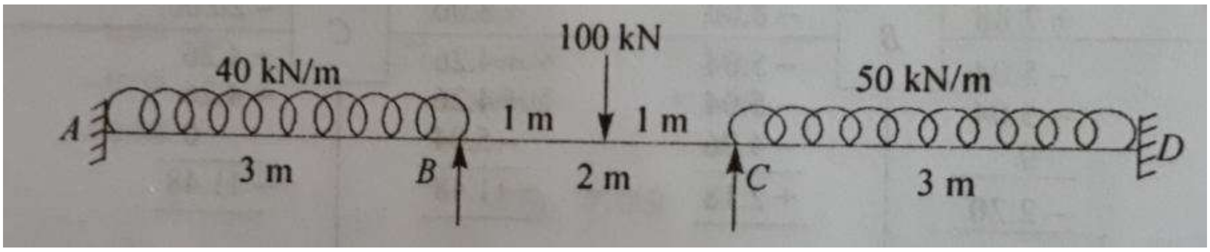
The quantity $m'_{ab} = m'_{ba}$ *is called diaplacement contribution of member AB*

Then

$$m_{ab} = \mu_{ab}\left[\sum \acute{M}_{ab} + \sum m_{ba} + \sum m'_{ab}\right]$$

Where $\mu_{ab} = \left(\frac{-1}{2}\frac{k_{ab}}{\sum k_{ab}}\right)$

Ex: **Determine the support moments for the continuous beam shown in figure using Kani's method. If Support B sinks by 2.5mm; for all members take I = 3.5**$\times 10^7 mm^4$ **and E = 200 KN/mm**



Sol:

$$EI = \frac{3.5 \times 10^7}{10^{12}} \times 200 \times 10^6 \, kN - m^2 = 7000 \, kN - m^2$$

$$\delta = 2.5 \, mm = 2.5 \times 10^{-3} \, m$$

For member AB

$$m'_{ab} = m'_{ba} = \frac{-6EI\delta}{L^2} = \frac{-6 \times 7000 \times 2.5 \times 10^{-3}}{9} = -11.67 \, kN - m$$

$$m'_{bc} = m'_{cb} = \frac{6EI\delta}{L^2} = \frac{-6 \times 7000 \times 2.5 \times 10^{-3}}{4} = 26.25 \, kN - m$$

$$\acute{M}_{ab} = \frac{-40 \times 3^2}{12} - 11.67 = -41.67 \, kN - m$$

$$\acute{M}_{ba} = \frac{40 \times 3^2}{12} - 11.67 = 18.33 \, kN - m$$

$$\acute{M}_{bc} = \frac{-Wl}{8} = \frac{-100 \times 2}{8} + 26.25 = 1.25 \, kN - m$$

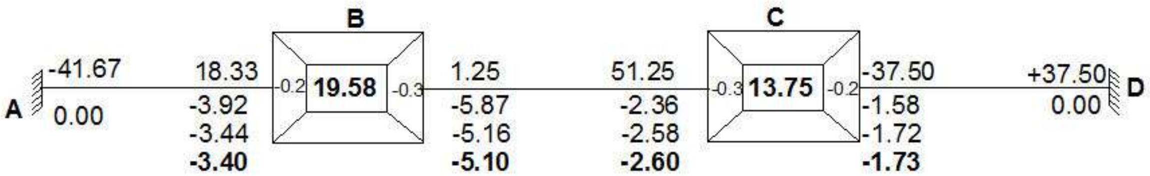
$$\acute{M}_{cb} = \frac{Wl}{8} = \frac{100 \times 2}{8} + 26.25 = 51.25 \, kN - m$$

$$\acute{M}_{cd} = \frac{-50 \times 3^2}{12} = -37.5 \, kN - m$$

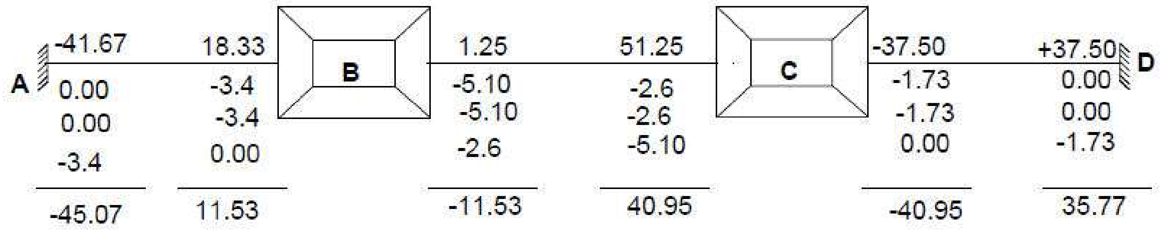
$$\acute{M}_{dc} = 37.5 \, kN - m$$

Rotation factors at joint B and C:

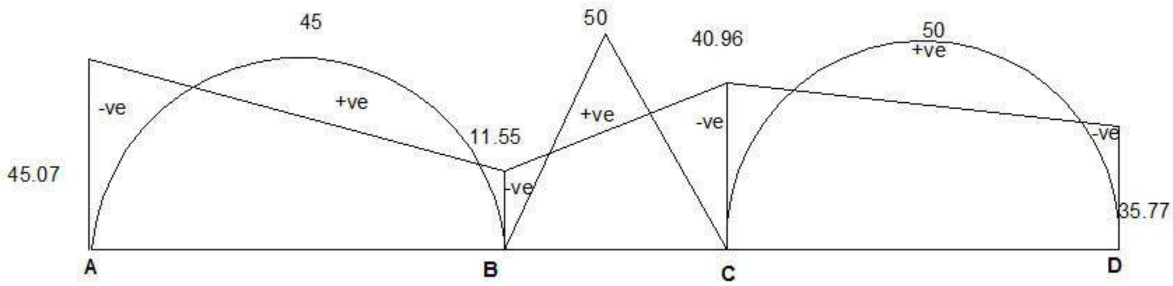
Joint	Member	Relative Stiffness (k)	Total Stiffness ($\sum k$)	Rotation factor(μ)
B	BA	I/3 = 2I/6	5I/6	-0.2
	BC	I/2 = 3I/6		-0.3
C	CB	I/2 = 3I/6	5I/6	-0.3
	CD	I/3 = 2I/6		-0.2



Final Moments:



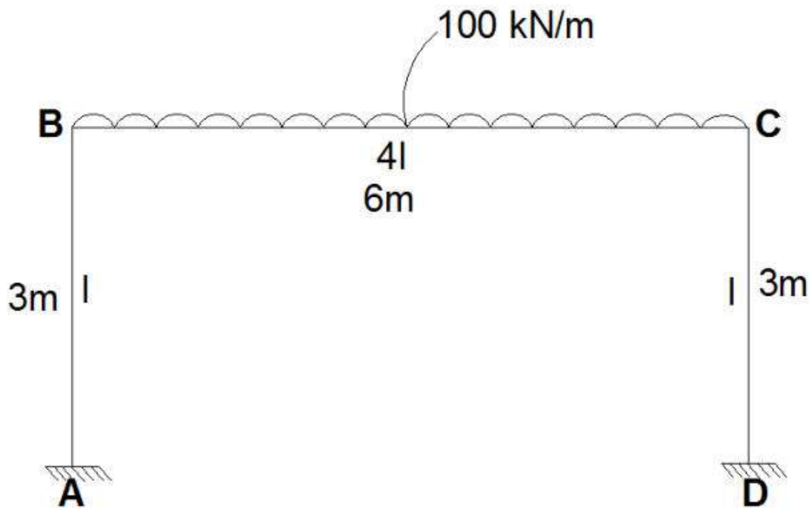
Bending Moment Diagram:



Portal frames without lateral sway:

When a portal frame is with symmetrically provided vertical and horizontal members and carries symmetrical vertical loading and symmetrical end condition, the frame will not undergo any lateral sway.

Ex: **Determine the moments at A, B, C and D for a portal frame loaded as shown in figure by Kani's method.**



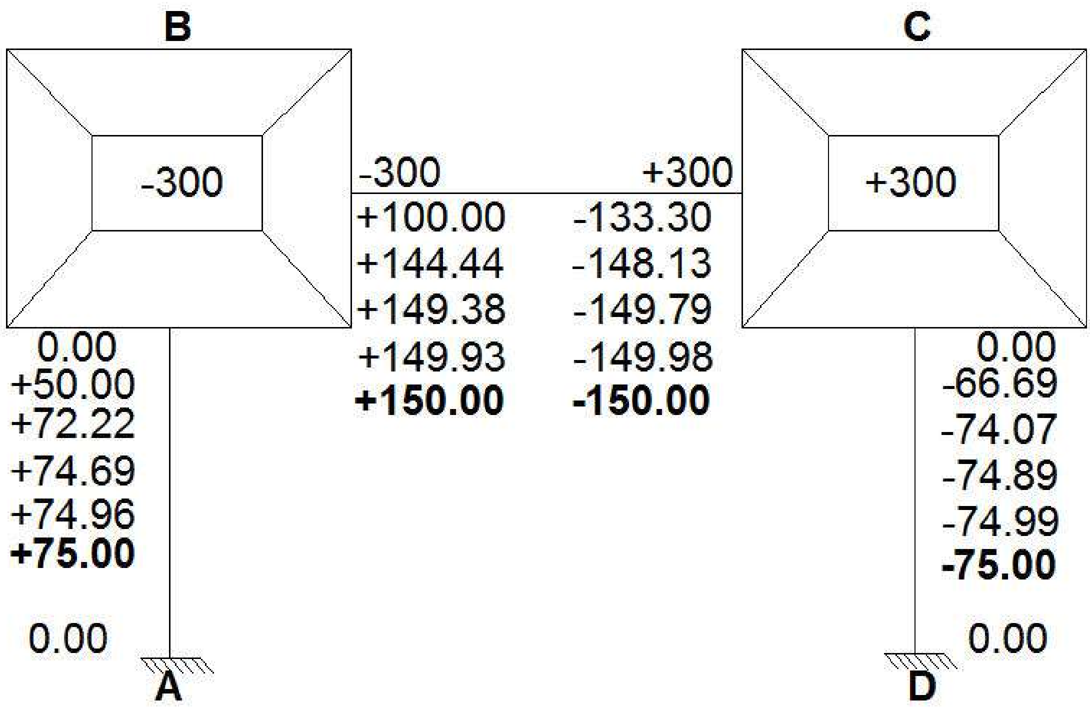
$$\acute{M}_{ab} = \acute{M}_{ba} = \acute{M}_{cd} = \acute{M}_{dc} = 0$$

$$\acute{M}_{bc} = \frac{-wl^2}{12} = \frac{-100 \times 6^2}{12} = -300\,kN - m$$

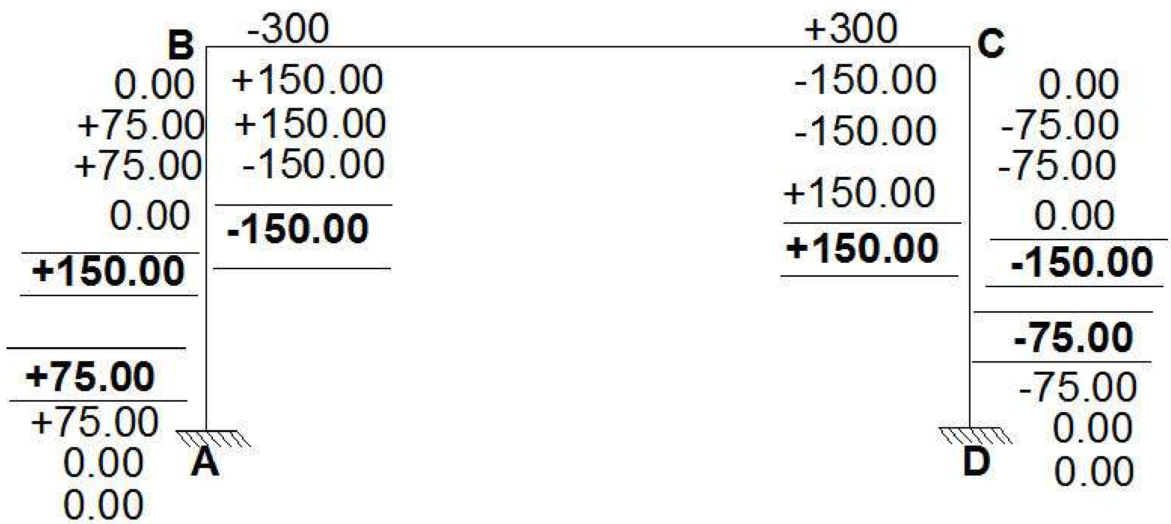
$$\acute{M}_{cb} = \frac{wl^2}{12} = \frac{100 \times 6^2}{12} = +300\,kN - m$$

Free moments in the span BC = $\frac{wl^2}{8} = \frac{100 \times 6^2}{8} = 450\,kN - m$

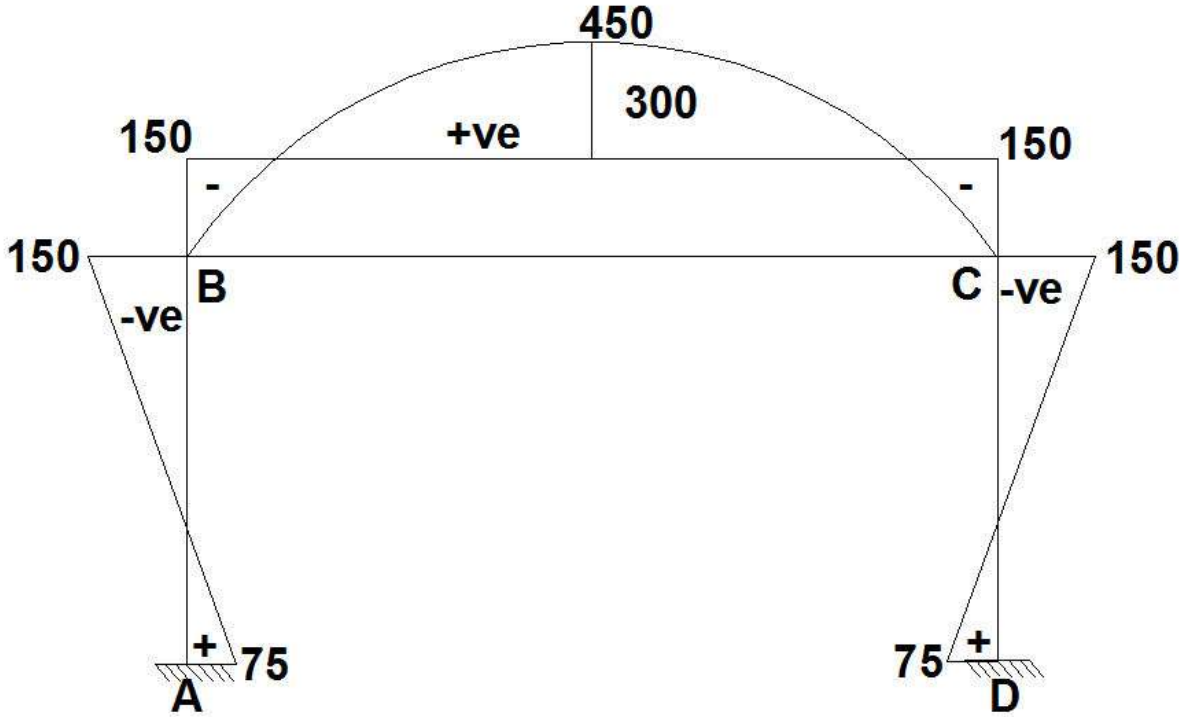
Joint	Member	Relative Stiffness (k)	Total Stiffness $(\sum k)$	Rotation factor $\mu_{ab} = \left(\frac{-1}{2}\frac{k_{ab}}{\sum k_{ab}}\right)$
B	BA	I/3 = 2I/6	6I/6	-1/6
	BC	4I/6		-1/3
C	CB	4I/6	6I/6	-1/3
	CD	2I/6		-1/6



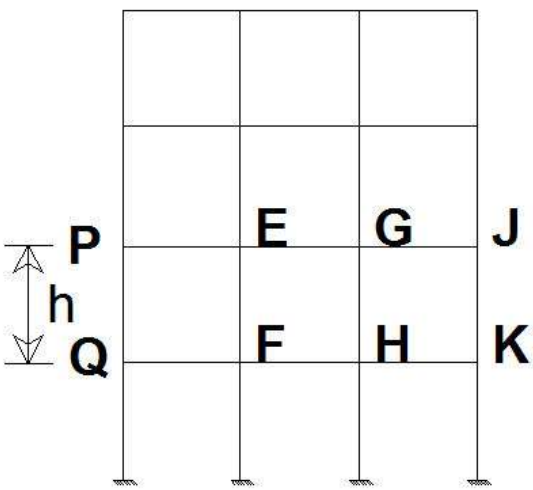
Final Moments:

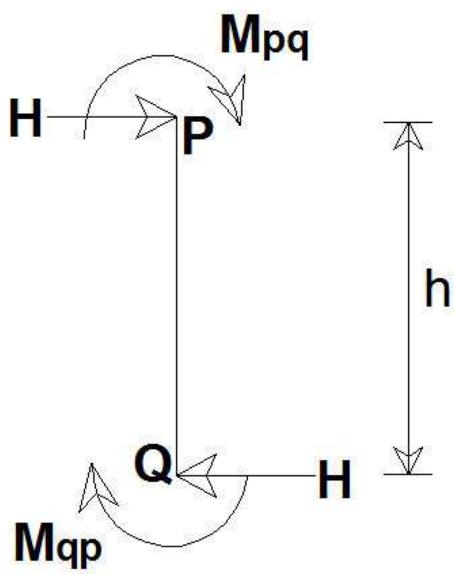


Bending Moment Diagram:



Analysis of frames with lateral sway due to vertical loading:





Let PQ is vertical member in r^th storey of a frame.

$Let\ M_{pq} \wedge M_{qp}\ be\ the\ end\ moments\ at\ P \wedge Q$

Let H – Horizontal forces extend by frame on member PQ at P and Q respectively.

From condition of equilibrium of member PQ,

$M_{pq}+M_{qp}+H\,h=0$

$H=\frac{-(M¿¿pq+M_{qp})}{h}¿$ ________________________(1) [Shear force in member PQ]

Let PQ, EF, GH and JK be the vertical members in rth storey,

$\sum H=-¿¿$

Let Sr - Sum of storey shear in columns of rth storey

$S_r=\sum H=-¿¿$

Where

$\sum M_{pq}=\sum \textit{of end moments at upper ends of all columns} \in r^{th} \textit{storey}$

$\sum M_{qp}=\sum \textit{of end moments at lower ends of all columns} \in r^{th} \textit{storey}$

Lateral sway due to vertical loading:

Since external loading is only due to vertical load.

Storey shear, $S_r=0$

$\sum M_{pq}+\sum M_{qp}=0¿$

Considering displacement contributions,

The general expressions for final end moments in PQ

$M_{pq}=\acute{M}_{pq}+2m_{pq}+m_{qp}+m'_{pq}$ ______________________________(4)

$M_{qp}=\acute{M}_{qp}+2m_{qp}+m_{pq}+m'_{pq}$

Where

$\acute{M}_{pq} \wedge \acute{M}_{qp}$ *are ¿ end moments at* $P \wedge Q$ *respectively*

m_{pq} *– Rotation contribution of end P*

m_{qp} *– Rotation contribution of end Q*

m'_{pq} *– Displacement contribution of PQ*

Since the loading is vertical only,

$\acute{M}_{pq}=\acute{M}_{qp}=0$

Therefore Equation (4) becomes

$M_{pq} = 2m_{pq} + m_{qp} + m'_{pq}$

$M_{qp} = m_{pq} + 2m_{qp} + m'_{pq}$

$M_{pq} + M_{qp} = 3m_{pq} + 3m_{qp} + 2m' ¿$

From Equation (3)

$\sum M_{pq} + \sum M_{qp} = 0$

$\sum M_{pq} + \sum M_{qp} = 3\sum m_{pq} + 3\sum m_{qp} + 2\sum m'_{pq} = 0$

$\sum m'_{pq} = \frac{-3}{2}\left[\sum m_{pq} + \sum m_{qp}\right] ¿$

The above condition is the relation between rotation contribution and displacement contribution.

For any member displacement contribution $¿ \mp \frac{6EI\delta}{L^2}$

Now consider all columns in rth storey

Let $\delta = Relative\ displacement\ of\ r^{th}\ storey$

$¿ \delta\ is\ samre\ for\ all\ columns$

Let L-Length of column and E-Young's Modulus

In $\mp \frac{6EI\delta}{L^2}, E \wedge \delta\ are\ assumed\ ¿\ be\ same\ for\ all\ columns$

$m'_{pq} \propto I, But\ relative\ stiffness\ k = \frac{I}{L}$

Since Length of column is same for all

$m'_{pq} \propto k$

$\frac{m'_{pq}}{\sum m'_{pq}} = \frac{k_{pq}}{\sum k_{pq}}$

Or

$m'_{pq} = \frac{k_{pq}}{\sum k_{pq}} \sum m'_{pq}$ ________________________ (7)

Substituting (6) in (7)

$$m'_{pq} = \frac{k_{pq}}{\sum k_{pq}}(-3/2)\left[\sum m_{pq} + \sum m_{qp}\right] \qquad (8)$$

The quantity $\frac{k_{pq}}{\sum k_{pq}}\left(\frac{-3}{2}\right) = u \text{ is called Displacement factor}$ of member **PQ**

Where

$\left[\sum m_{pq} + \sum m_{qp}\right]$ *represents* $\sum$ *of rotation contributions of top* $\wedge$ *bottom ends of all columns* $\in r^{th}$ *storey*

$\sum k_{pq} = \sum$ *of relative stiffness of all columns* $\in r^{th}$ *storey*

Sum of displacement factors of all columns in a storey = -3/2

Final moments can be obtained as follows

$(a)\, M_{pq} = \acute{M}_{pq} + 2m_{pq} + m_{qp} + m'_{pq}$

$(b)\, m_{pq} = \mu_{pq}\left[\sum \acute{M}_{pq} + \sum m_{qp} + \sum m_{pq}\right]$

$(c)\, m'_{pq} = \frac{k_{pq}}{\sum k_{pq}}\left(\frac{-3}{2}\right)\left[\sum m_{pq} + \sum m_{qp}\right]$ *for a storey*

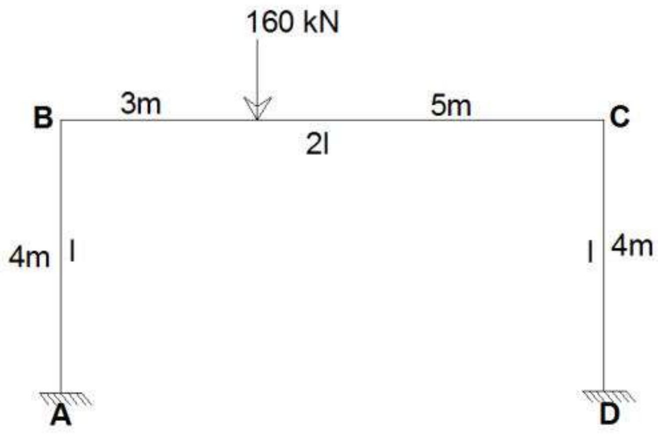
Procedure for Kani's distribution:

1. Find the fixed end moments in all the members
2. Find the rotation factors at all the joints which are going to rotate
3. Find displacement factors for all the columns in each storey
4. Prepare Kani's distribution table
5. Initially, all the rotation contributions and displacement contributions are assumed to be zero
6. As and when rotation contributions are available, those values are considered in the analysis.
7. Kani's procedure is applied joint by joint to calculate rotation contributions till all the of the frame are completed in the 1st cycle
8. Before going for next cycle, displacement contributions of all columns in each storey are calculated using equation (8)
9. In second cycle, displacement contributions should also be considered while calculating the rotation contributions since they are available using equation 9(b)
10. Repeat the cycle till the rotation and displacement contributions are almost same with the values of previous cycle
11. Assemble the final moments using equation 9(a)

Note:

For vertical load analysis, there are no displacement contributions in beam and no fixed end moments for columns.

Ex: **Determine the moments at A, B, C and D for portal frame shown in figure using Kani's method.**



Sol:

$$\acute{M}_{ab}=\acute{M}_{ba}=\acute{M}_{cd}=\acute{M}_{dc}=0$$

$$\acute{M}_{bc}=\frac{-W\,ab^2}{L^2}=\frac{-160\times 3\times 5^2}{8^2}=-187.50\,kN-m$$

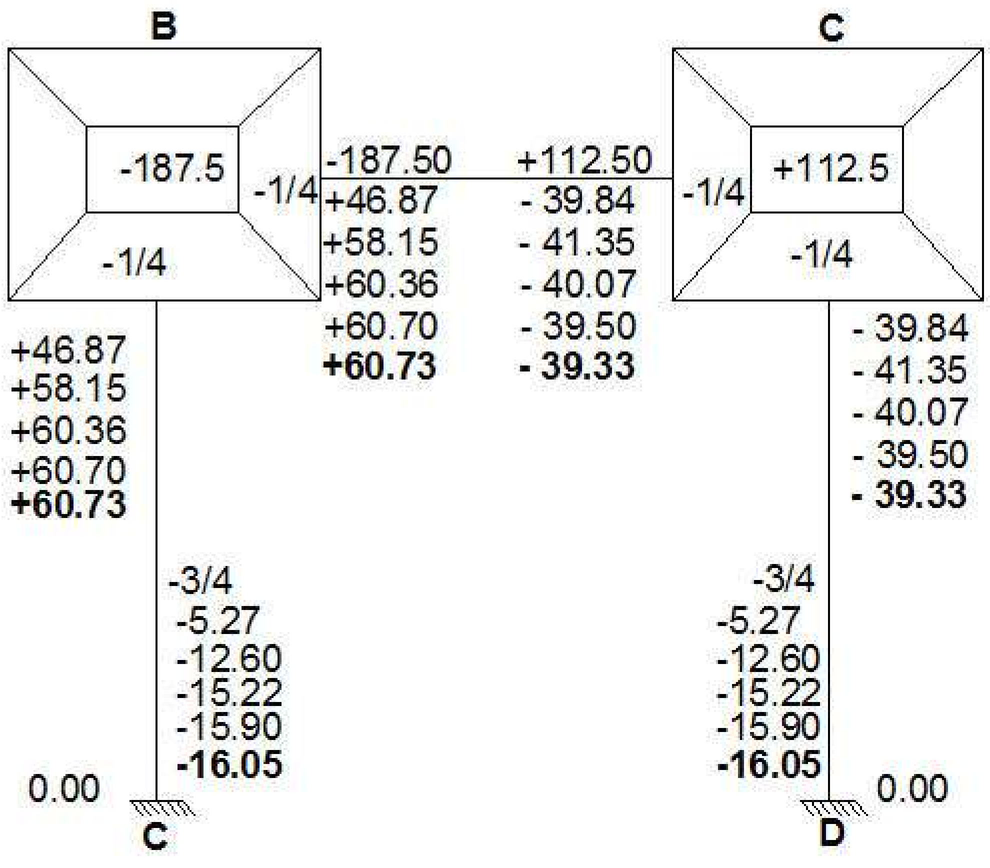
$$\acute{M}_{cb}=\frac{W\,a^2 b}{L^2}=\frac{160\times 5\times 3^2}{8^2}=112.50\,kN-m$$

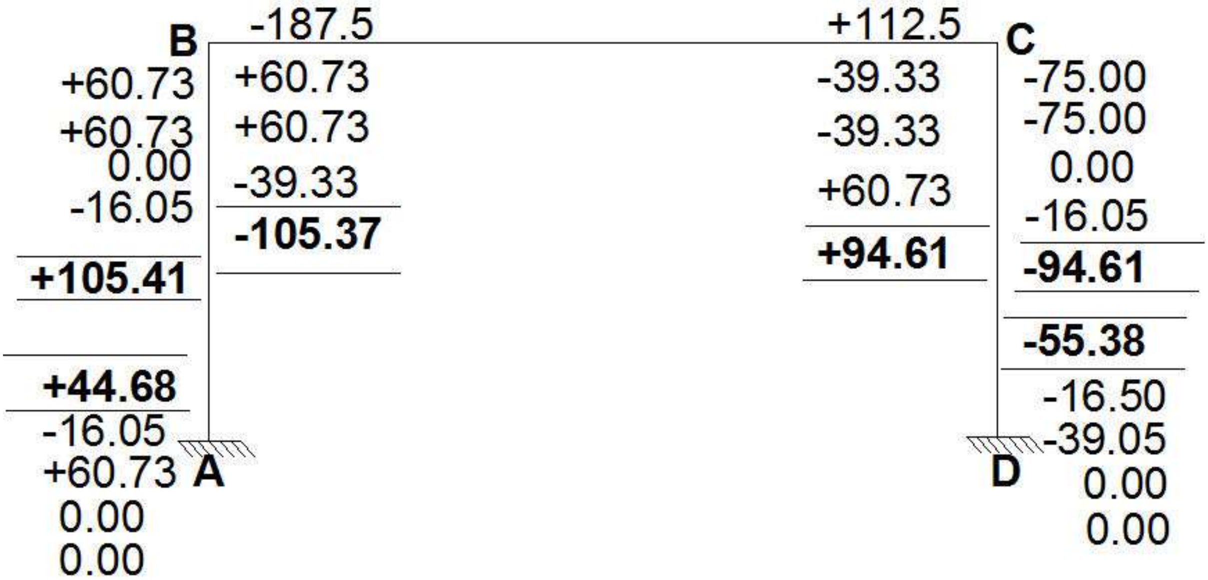
Rotation factors:

Joint	Member	Relative Stiffness (k)	Total Stiffness $(\sum k)$	Rotation factor $\mu_{ab}=\left(\frac{-1}{2}\frac{k_{ab}}{\sum k_{ab}}\right)$
B	BA	I/4 = 2I/6	2I/4	-1/4
	BC	2I/8 = I/4		-1/4
C	CB	I/4	2I/4	-1/4
	CD	I/4		-1/4

Displacement factor:

Joint	Relative Stiffness (k)	Total Stiffness ($\sum k$)	Rotation factor $\mu_{ab}=\left(\frac{-3}{2}\frac{k_{ab}}{\sum k_{ab}}\right)$
AB	I/4	2I/4	-3/4
CD	I/4		-3/4





$$M_{pq} = M'_{pq} + 2m_{pq} + m_{qp} + m'_{pq}$$

$$m'_{pq} = \frac{k_{pq}}{\sum k_{pq}} \left(\frac{-3}{2}\right) \left[\sum m_{pq} + \sum m_{qp}\right] \text{for a storey}$$

1st cycle:

Rotation contributions:

At joint A,

$$m_{bc} = \left(\frac{-1}{4}\right)[-187.5 + 0 + 0 + 0] = +46.87$$

$$m_{ba} = \left(\frac{-1}{4}\right)[-187.5 + 0 + 0 + 0] = +46.87$$

At joint B,

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 46.87 + 0] = -39.84$$

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 46.87 + 0] = -39.84$$

Displacement Contributions:

Storey one:

$$m'_{ab} = \left(\frac{-3}{4}\right)[46.85 - 39.33 + 0] = -5.27$$

$$m'_{cd} = \left(\frac{-3}{4}\right)[46.85 - 39.33 + 0] = -5.27$$

2nd cycle:

Rotation contributions:

At joint A,

$$m_{bc} = \left(\frac{-1}{4}\right)[-187.5 + 0 - 39.84 + 0 - 5.27] = +58.5$$

$$m_{ba} = \left(\frac{-1}{4}\right)[-187.5 + 0 - 39.84 + 0 - 5.27] = +58.5$$

At joint B,

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 46.87 + 0] = -41.35$$

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 46.87 + 0] = -41.35$$

Displacement Contributions:

Storey one:

$$m'_{ab} = \left(\frac{-3}{4}\right)[58.15 - 41.35 + 0 + 0] = +12.60$$

$$m'_{cd} = \left(\frac{-3}{4}\right)[58.15 - 41.35 + 0 + 0] = +12.60$$

3rd cycle:

Rotation contributions:

At joint A,

$$m_{bc} = \left(\frac{-1}{4}\right)[-187.5 + 0 - 41.35 - 12.60] = +60.36$$

$$m_{ba} = \left(\frac{-1}{4}\right)[-187.5 + 0 - 41.35 - 12.60] = +60.36$$

At joint B,

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 60.36 - 12.60] = -40.07$$

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 60.36 - 12.60] = -40.07$$

Displacement Contributions:

Storey one:

$$m'_{ab} = \left(\frac{-3}{4}\right)[60.36 - 40.07 + 0 + 0] = -15.22$$

$$m'_{cd} = \left(\frac{-3}{4}\right)[60.36 - 40.07 + 0 + 0] = -15.22$$

4th cycle:

Rotation contributions:

At joint A,

$$m_{bc} = \left(\frac{-1}{4}\right)[-187.5 - 40.07 - 15.22] = +60.70$$

$$m_{ba} = \left(\frac{-1}{4}\right)[-187.5 - 40.07 - 15.22] = +60.70$$

At joint B,

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 60.70 - 15.22] = -39.5$$

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 60.70 - 15.22] = -39.5$$

Displacement Contributions:

Storey one:

$$m'_{ab} = \left(\frac{-3}{4}\right)[60.70 - 39.5 + 0 + 0] = -15.90$$

$$m'_{cd} = \left(\frac{-3}{4}\right)[60.70 - 39.5 + 0 + 0] = -15.90$$

5th cycle:

Rotation contributions:

At joint A,

$$m_{bc} = \left(\frac{-1}{4}\right)[-187.5 - 39.5 - 15.90] = +60.73$$

$$m_{ba} = \left(\frac{-1}{4}\right)[-187.5 - 39.5 - 15.90] = +60.73$$

At joint B,

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 60.73 - 15.90] = -39.33$$

$$m_{cb} = \left(\frac{-1}{4}\right)[+112.5 + 60.73 - 15.90] = -39.33$$

Displacement Contributions:

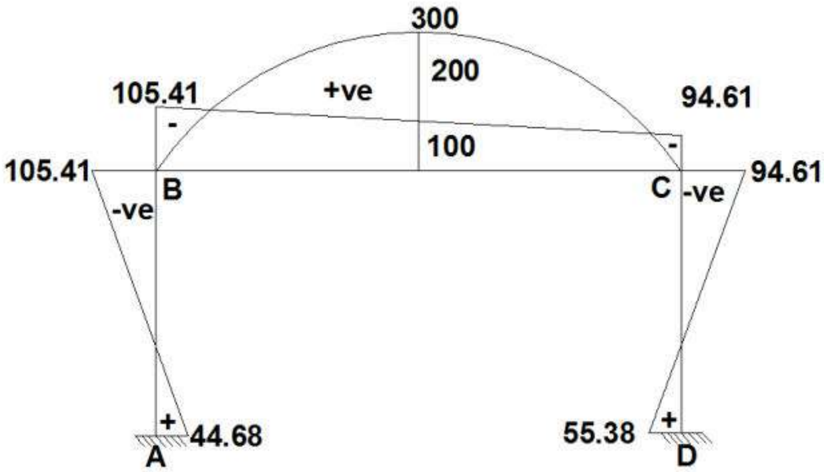
Storey one:

$$m'_{ab} = \left(\frac{-3}{4}\right)[60.73 - 39.33 + 0 + 0] = -16.05$$

$$m'_{cd} = \left(\frac{-3}{4}\right)[60.73 - 39.33 + 0 + 0] = -16.05$$

Free moment of AB = $\frac{Wab}{L}$ = $\frac{160 \times 3 \times 5}{8} = +300\, kN - m$

Bending Moment Diagram:



Theory of Structures

III year-I Semester

Unit-VI

Matrix Methods

Learning Material

Matrix analysis of structures:

Definition of flexibility and stiffness influence coefficients – development of flexibility matrices by physical approach energy principle.
These are the two basic methods by which an indeterminate skeletal structure is analyzed. In these methods flexibility and stiffness properties of members are employed. These methods have been developed in conventional and matrix forms. Here conventional methods are discussed

Flexibility Method:

The given indeterminate structure is first made statically determinate by introducing suitable number of releases. The number of releases required is equal to statical indeterminacy $\propto s$. Introduction of releases results in displacement discontinuities at these releases under the externally applied loads. Pairs of unknown biactions (forces and moments) are applied at these releases in order to restore the continuity or compatibility of structure. The computation of these unknown biactions involves solution of linear simultaneous equations. The number of these equations is equal to static indeterminacy $\propto s$. After the unknown biactions are computed all the internal forces can be computed in the entire structure using equations of equilibrium and free bodies of members. The required displacements can also be computed using methods of displacement computation.

In flexibility method since unknowns are forces at the releases the method is also called force method. Since computation of displacement is also required at releases for imposing conditions of compatibility the method is also called compatibility method. In computation of displacements use is made of flexibility properties, hence, the method is also called flexibility method.

Stiffness Method:

The given indeterminate structure is first made kinematically determinate by introducing constraints at the nodes. The required number of constraints is equal to degrees of freedom at the nodes that is kinematic indeterminacy $\propto k$. The kinematically determinate structure comprises of fixed ended members, hence, all nodal displacements are zero. These results in stress resultant discontinuities at these nodes under the action of applied loads or in other words the clamped joints are not in equilibrium. In order to restore the equilibrium of stress resultants at the nodes the nodes are imparted suitable unknown displacements. The number of simultaneous equations representing joint equilibrium of forces is equal to kinematic indeterminacy ∝k. Solution of these equations gives unknown nodal displacements. Using stiffness properties of members the member end forces are computed and hence the internal forces throughout the structure.

Since nodal displacements are unknowns, the method is also called displacement method. Since equilibrium conditions are applied at the joints the method is also called equilibrium method. Since stiffness properties of members are used the method is also called stiffness method.

The force method involves five steps. They are briefly mentioned here; but they are explained future in examples and in sections below.

1. First of all, the degree of statical indeterminacy is determined. A number of releases equal to the degree of indeterminacy is now introduced, each release being made by the removal of an external or an internal force. The releases must be chosen so that the remaining structure is stable and statically determinate. However , we will learn that in some case the number of releases can be less than the degree of indeterminacy , provided the remaining statically indeterminate structure is so simple that it can be readily analyzed. In all cases, the released forces, which are also called redundant forces, should be carefully chosen so that the released structure is easy to analyze.

2. Application of the given loads on the released structure will produce displacements that are inconsistent with the actual structure, such as a rotation or a translation at a support where this displacement must be zero. In the second step these inconsistencies or "errors" in the released structure are determined. In other words, we calculate the magnitude of the "errors" in the displacements corresponding to the redundant forces. These displacements may be due to external applied loads, settlement of supports, or temperature variation.

3. The third step consists of a determination of the displacements in the released structure due to unit values of the redundants. These displacements are required in step 2.

4. The values of the redundant forces necessary to eliminate the errors in the displacements are now determined. This requires the writing of super position equations in which the effects of the separate redundants are added to the displacements of the released.

5. Hence, we find the forces on the original indeterminate structure: they are the sum of the correction forces (redundants) and forces on the released structure.

The displacement method involves five steps:

1.First of all, the degree of kinematic indeterminacy has to be found. A coordinate system is then established to identify the location and direction of the joint displacements. Restraining forces equal in number to the degree of kinematic indeterminacy are introduced at the coordinates to prevent the displacement of the joints. In some cases, the number of restraints introduced may be smaller than the degree of kinematic indeterminacy, provided that the analysis of the resulting structure is a standard one is therefore known.

We should note that, unlike the force method, the above procedure requires no choice to be made with respect to the restraining forces. This fact favours the use of the displacement method in general computer programs for the analysis of a structure.

2. The restraining forces are now determined as a sum of the fixed-end forces for the members meeting at a joint. For most practical cases, the fixed-end forces can be calculated with the aid of standard tables . An external force at a coordinate is restrained simply by an equal and opposite force that must be added to the sum of the fixed-end forces.

We should remember that the restraining forces are those required to prevent the displacement at the coordinates due to all effects, such as external loads, temperature variation, or restrain. These effects may be considered separately or may be combined.

If the analysis is to be performed for the effect of movement of one of the joints in the structure, for example, the settlement of a support, the forces at the coordinates required to hold the joint in the displaced position are included in the restraining forces.

The internal forces in the members are also determined at the required locations with the joints in the restrained position

3. The structure is now assumed to be deformed in such a way that a displacement at one of the coordinates equals unity and all the other displacements are zero, and the forces required to hold the structure in this configuration are determined. These forces are applied at the coordinates representing the degrees of freedom. The internal forces at the required locations corresponding to this configuration are determined.

The process is repeated for a unit value of displacement at each of the coordinates separately.

4. The values of the displacements necessary to eliminate the restraining forces introduces in are determined. This requires superposition equations in which the effects of separate displacements on the restraining forces are added.

5. Finally, the forces on the original structure are obtained by adding the forces on the restrained structure to the forces caused by the joint displacements will be determined.

Flexibility and Stiffness:

Flexibility and its converse, known as *stiffness*, are important properties which characterize the response of a structure by means of the force-displacement relationship. In a general sense, the flexibility of a structure is defined as the displacement caused by a unit force and the stiffness is defined as the force required for a unit displacement. Consider first, a structural element with a single degree of freedom. The spring *AB*, shown

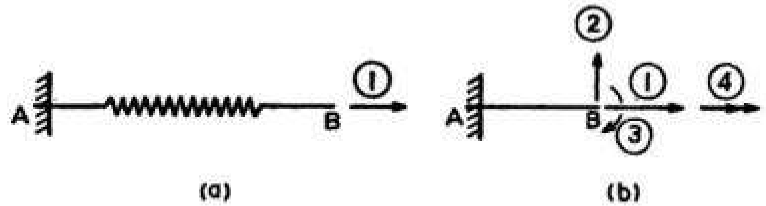


FIG. 4.1

in Fig. 4.1(a), is fixed at end *A* and has a single degree of freedom at end *B* along coordinate 1. The flexibility of the spring is defined as the displacement δ_{11} at coordinate 1 due to a unit force at coordinate 1. If a force P_1 produces a displacement Δ_1 at coordinate 1,

$$\text{flexibility} = \frac{\Delta_1}{P_1} = \delta_{11} \tag{4.1}$$

Similarly, the stiffness of the spring is defined as the force k_{11} required for a unit displacement at coordinate 1.

$$\text{stiffness} = \frac{P_1}{\Delta_1} = k_{11} \tag{4.2}$$

Consider next, a structural element with multiple degrees of freedom. The structural member *AB* of uniform cross-section, shown in Fig. 4.1(b),

is fixed at end *A*. End *B* can have the following four types of displacements:

(i) axial displacement Δ_1 at coordinate 1,
(ii) transverse displacement Δ_2 at coordinate 2,
(iii) bending or flexural displacement Δ_3 at coordinate 3 and
(iv) torsional displacement or twist Δ_4 at coordinate 4.

The flexibility and stiffness of structural member *AB*, with respect to each of the four types of displacements, may now be defined as follows:

4.1.1 Axial Displacement

If an axial force P_1 is applied at coordinate 1, displacement Δ_1 at coordinate 1 is given by the equation

$$\Delta_1 = \frac{P_1 L}{AE} \tag{4.3}$$

where L = length of the member

A = cross-sectional area of the member

E = modulus of elasticity.

As flexibility is the displacement caused by a unit force, the flexibility with respect to axial displacement is obtained by putting $P_1 = 1$ in Eq. (4.3).

$$\text{axial flexibility, } \delta_{11} = \frac{L}{AE} \tag{4.4}$$

By definition, the axial stiffness of the member is the force required for unit displacement along coordinate 1. Hence, putting $\Delta_1 = 1$ in Eq. (4.3).

$$\text{axial stiffness, } k_{11} = \frac{AE}{L} \tag{4.5}$$

The flexibility and stiffness with respect to axial displacement given by Eqs. (4.4) and (4.5) are of relevance to members of pin-jointed frames which carry axial forces only. In the case of rigid-jointed frames, the axial displacements are small as compared to transverse displacements. Consequently, it is a common practice in the analysis of rigid-jointed frames to ignore the axial flexibility of the member. In other words, the members of the rigid-jointed frames are considered to be infinitely stiff with respect to axial displacements.

4.1.2 Transverse Displacement

It has been shown in Sec. 2.14 that force P_2 required at coordinate 2 for displacement Δ_2 at coordinate 2 without any displacement at coordinates 1, 3 and 4 is given by the equation

$$P_2 = \frac{12EI\Delta_2}{L^3} \tag{4.6}$$

Hence, by definition, the flexibility and stiffness with respect to transverse displacement may be written as

$$\text{transverse flexibility, } \delta_{22} = \frac{L^3}{12EI} \tag{4.7}$$

and

$$\text{transverse stiffness, } k_{22} = \frac{12EI}{L^3} \tag{4.8}$$

Equations (4.7) and (4.8) are based on the assumption that end A, known as the far-end, is fixed. If far-end A is hinged, the force P_2 required at coordinate 2 for a displacement Δ_2 at coordinate 2 without any displacement at coordinates 1, 3 and 4 is given by Eq. (2.45b)

$$P_2 = \frac{3EI\Delta_2}{L^3} \tag{4.9}$$

Hence, by definition, the flexibility and stiffness with respect to transverse displacement may be written as

$$\text{transverse flexibility, } \delta_{22} = \frac{L^3}{3EI} \tag{4.10}$$

and

$$\text{transverse stiffness, } k_{22} = \frac{3EI}{L^3} \tag{4.11}$$

4.1.3 Bending or Flexural Displacement

It has been shown in Sec. 2.14 that the force P_3 required at coordinate 3 for displacement Δ_3 at coordinate 3 without any displacement at coordinates 1, 2 and 4 is given by the equation

$$P_3 = \frac{4EI\Delta_3}{L} \tag{4.12}$$

Hence, by definition, the flexibility and stiffness with respect to flexural displacement may be written as

$$\text{flexural flexibility, } \delta_{33} = \frac{L}{4EI} \tag{4.13}$$

and

$$\text{flexural stiffness, } k_{33} = \frac{4EI}{L} \tag{4.14}$$

Equations (4.13) and (4.14) are based on the assumption that far-end A is fixed. If far-end A is hinged, the force P_3 required at coordinate 3 for a displacement Δ_3 at coordinate 3 without any displacement at coordinates 1, 2 and 4 is given by Eq. (2.43a)

$$P_3 = \frac{3EI\Delta_3}{L} \tag{4.15}$$

Hence, by definition, the flexibility and stiffness with respect to flexural displacement may be written as

$$\text{flexural flexibility, } \delta_{33} = \frac{L}{3EI} \tag{4.16}$$

and

$$\text{flexural stiffness, } k_{33} = \frac{3EI}{L} \tag{4.17}$$

S. No.	*Type of displacement*, Δ	*Flexibility*, δ	*Stiffness*, k
1.	Axial	$\frac{L}{AE}$	$\frac{AE}{L}$
2.	Transverse		
	(a) Far-end fixed	$\frac{L^3}{12EI}$	$\frac{12EI}{L^3}$
	(b) Far-end hinged	$\frac{L^3}{3EI}$	$\frac{3EI}{L^3}$
3.	Bending or flexural		
	(a) Far-end fixed	$\frac{L}{4EI}$	$\frac{4EI}{L}$
	(b) Far-end hinged	$\frac{L}{3EI}$	$\frac{3EI}{L}$
4.	Torsional	$\frac{L}{GK}$	$\frac{GK}{L}$

Flexibility matrix method:

Consider a structure which satisfies the basic assumptions enumerated in Sec. 2.2. Let the system of forces $P_1, P_2, \ldots, P_n$ act on the structure. The word 'forces' has been used here in the generalized sense so as to include couples and reaction components. The system of forces $P_1, P_2, \ldots, P_n$ may include all or some of the forces acting on the structure. Let the system of forces $P_1, P_2, \ldots, P_n$ produce displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ at coordinates

1, 2, ..., *n*. Using the principle of superposition discussed in Sec. 2.2, displacements $\Delta_1, \Delta_2, \ldots, \Delta_n$ may be expressed by the equations

$$\begin{aligned}
\Delta_1 &= \delta_{11}P_1 + \delta_{12}P_2 + \ldots + \delta_{1j}P_j + \ldots + \delta_{1n}P_n \\
\Delta_2 &= \delta_{21}P_1 + \delta_{22}P_2 + \ldots + \delta_{2j}P_j + \ldots + \delta_{2n}P_n \\
&\vdots \\
\Delta_i &= \delta_{i1}P_1 + \delta_{i2}P_2 + \ldots + \delta_{ij}P_j + \ldots + \delta_{in}P_n \\
&\vdots \\
\Delta_n &= \delta_{n1}P_1 + \delta_{n2}P_2 + \ldots + \delta_{nj}P_j + \ldots + \delta_{nn}P_n
\end{aligned} \qquad (4.21)$$

In Eq. (4.21), δ_{ij} is the displacement at coordinate i due to a unit force at coordinate j. Hence, $\delta_{i1}P_1$ is the displacement at coordinate i due to P_1. Similarly, $\delta_{i2}P_2$ is the displacement at coordinate i due to P_2. Hence, the total displacement at coordinate i due to all the forces may be expressed as

$$\Delta_i = \delta_{i1}P_1 + \delta_{i2}P_2 + \ldots + \delta_{in}P_n$$

This equation is the same as Eq. (2.5). This explains how Eq. (4.21) have been written down. As explained in Sec. 3.6, the set of simultaneous Eq. (4.21), representing the *force-displacement relationship* may be expressed in the following matrix form:

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_i \\ \vdots \\ \Delta_n \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} \ldots \delta_{1j} \ldots \delta_{1n} \\ \delta_{21} & \delta_{22} \ldots \delta_{2j} \ldots \delta_{2n} \\ \vdots & \\ \delta_{i1} & \delta_{i2} \ldots \delta_{ij} \ldots \delta_{in} \\ \vdots & \\ \delta_{n1} & \delta_{n2} \ldots \delta_{nj} \ldots \delta_{nn} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_j \\ \vdots \\ P_n \end{bmatrix} \qquad (4.22)$$

Equation (4.22) may be written in the compact form

$$[\Delta] = [\delta][P] \qquad (4.23)$$

where $[\Delta]$ = a column matrix of order $n \times 1$, known as *displacement matrix*

$[P]$ = a column matrix of order $n \times 1$, known as *force matrix*

$[\delta]$ = a square matrix of order n, known as *flexibility matrix.*

From Eq. (4.22) it may be noted that the elements of the jth column of the flexibility matrix are the displacements at coordinates 1, 2, ..., n due to a unit force at coordinate j. Hence, in order *to generate the jth column of the flexibility matrix, a unit force should be applied at coordinate j and the displacements at all the coordinates determined. These displacements constitute the elements of the jth column of the flexibility matrix.* Hence, in order to develop the flexibility matrix, a unit force should be applied successively at coordinates 1, 2, ..., n and the displacements at all the coordinates computed.

Stiffness Matrix method:

Let 1, 2, ..., n be the system of coordinates chosen to express the system

of forces $P_1, P_2, ..., P_n$ producing displacements $\Delta_1, \Delta_2, ..., \Delta_n$. If a unit displacement is given at coordinate j without any displacement at other coordinates, the forces required at coordinates 1, 2, ..., n may be represented by $k_{1j}, k_{2j}, ..., k_{nj}$ respectively. These are the forces which must act at coordinates 1, 2, ..., n to hold the structure in this specific deformed position in which $\Delta_j = 1$ and $\Delta_i(i \neq j) = 0$. In other words, k_{1j}, k_{2j}, ..., k_{nj} are the forces required at coordinates 1, 2, ..., n respectively in order to produce a unit displacement at coordinate j and zero displacement at all other coordinates. Thus k_{ij} is the force at coordinate i due to a unit displacement at coordinate j only. The total force P_i at coordinate i due to displacements $\Delta_1, \Delta_2, ..., \Delta_n$ may be computed by using the principle of superposition, Sec. 2.2.

$$P_i = k_{i1}\Delta_1 + k_{i2}\Delta_2 + ... + k_{in}\Delta_n$$

This equation is the same as Eq. (2.6). Similar equations can be written for the forces at other coordinates resulting in the following set of simultaneous equations:

$$\left.\begin{aligned} P_1 &= k_{11}\Delta_1 + k_{12}\Delta_2 + ... + k_{1j}\Delta_j + ... + k_{1n}\Delta_n \\ P_2 &= k_{21}\Delta_1 + k_{22}\Delta_2 + ... + k_{2j}\Delta_j + ... + k_{2n}\Delta_n \\ &\vdots \\ P_i &= k_{i1}\Delta_1 + k_{i2}\Delta_2 + ... + k_{ij}\Delta_j + ... + k_{in}\Delta_n \\ &\vdots \\ P_n &= k_{n1}\Delta_1 + k_{n2}\Delta_2 + ... + k_{nj}\Delta_j + ... + k_{nn}\Delta_n \end{aligned}\right\} \quad (4.24)$$

Equation (4.24), representing the force-displacement relationship, may be expressed in the following matrix form:

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_i \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \ldots k_{1j} \ldots k_{1n} \\ k_{21} & k_{22} \ldots k_{2j} \ldots k_{2n} \\ \vdots & \\ k_{i1} & k_{i2} \ldots k_{ij} \ldots k_{in} \\ \vdots & \\ k_{n1} & k_{n2} \ldots k_{nj} \ldots k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_j \\ \vdots \\ \Delta_n \end{bmatrix} \tag{4.25}$$

Equation (4.25) may be written in the compact form

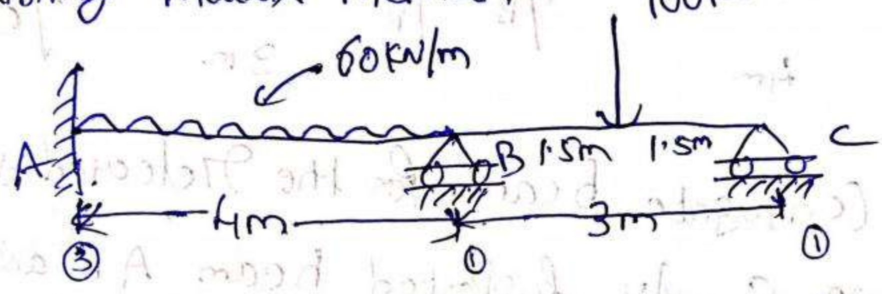
$$[P] = [k][\Delta] \tag{4.26}$$

where $[k]$ = a square matrix of order n, known as *stiffness matrix.*
From Eq. (4.25) it may be noted that the elements of the jth column of the stiffness matrix are the forces at coordinates 1, 2, ...,n due to a unit displacement at coordinate j. Hence, ***in order to generate the jth column of the stiffness matrix, a unit displacement must be given at coordinate j without any displacement at other coordinates and the forces required at all the co-ordinates determined. These forces constitute the elements of the jth column of the stiffness matrix.*** Hence, in order to develop the stiffness matrix, unit displacement should be given successively at coordinates 1, 2, ..., n and forces at all the coordinates calculated.

Flexibility method Example:

Flexibility matrix Method — 20/1/21

Problem (1):- Analyse the continuous beam by flexibility matrix method.



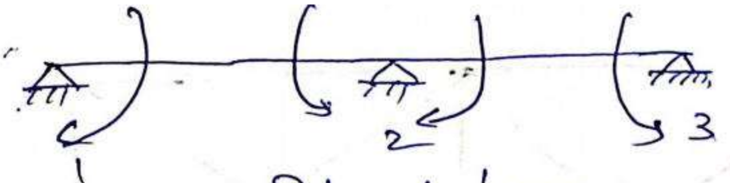
Solution :- Degree of Static indeterminacy

$D_S = R - E - A \Rightarrow D_S = 5 - 3 - 0$

$\boxed{D_S = 2}$ 2×2

Selecting M_A & M_B as the redundant forces.

∴ the released structures are the two independent simply supported beams AB and BC as shown in below figure.



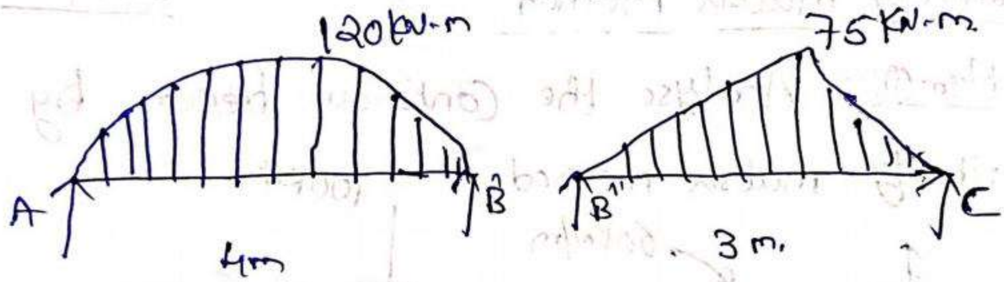
Released beam.

The bending moment diagram due to loads.

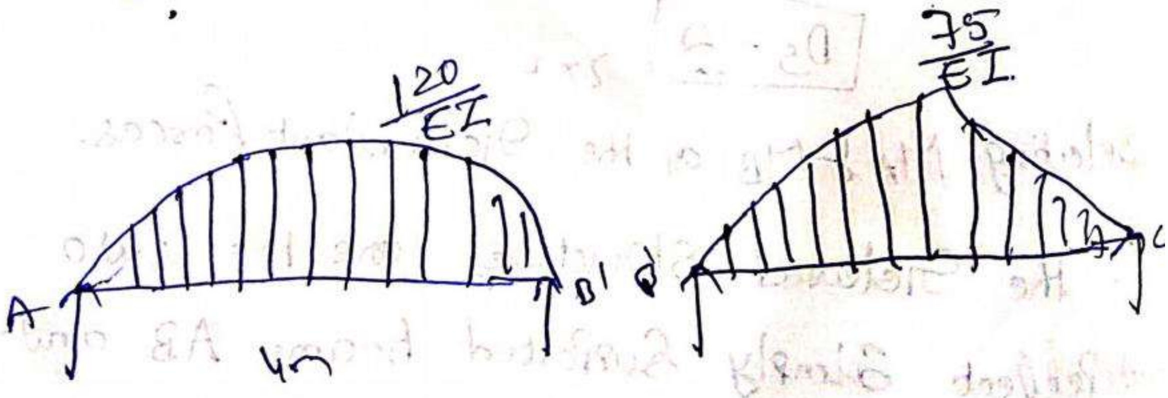
Free bending moment diagram.

$$M_{AB} = \frac{wl^2}{8} = \frac{60 \times 4^2}{8} = 120 \text{ kN.m.}$$

$$M_{BC} = \frac{wl}{4} = \frac{100 \times 3}{4} = 75 \text{ kN.m}$$



The conjugate beam for the released structure has two simply supported beam AB' and $B'C$ with $\frac{M}{EI}$ diagram.

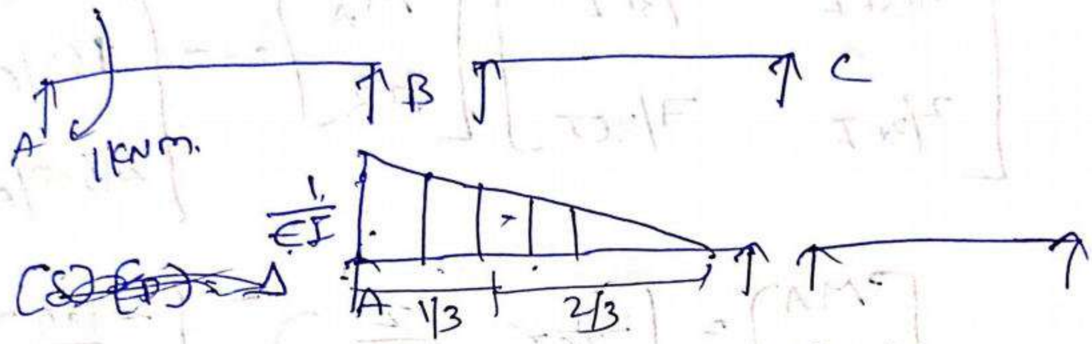


$$\Delta_{1L} = \frac{2}{3} \times \frac{120}{EI} \times 4 = \frac{160}{EI}$$

Similarly $\Delta_{2L} = A^0B^1 + B^1C$

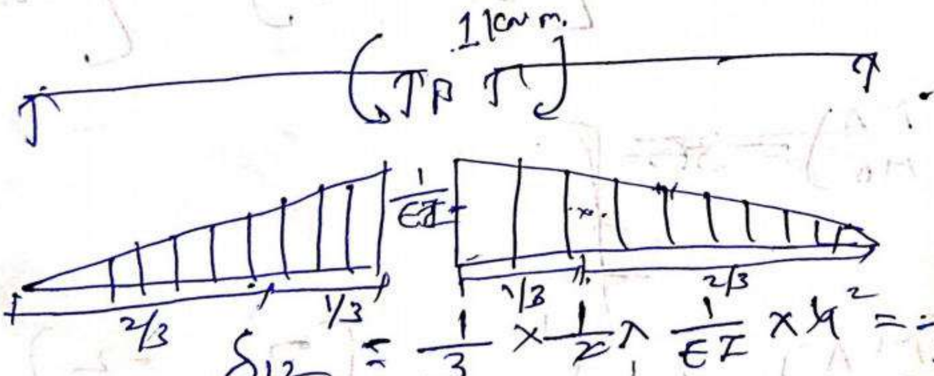
$$= \frac{\left(\frac{2}{3}\times\frac{120}{EI}\times 4\right)}{2} + \left(\frac{\frac{1}{2}\times\frac{75}{EI}\times 3}{2}\right)$$

$$\frac{160}{EI} + \frac{56.25}{EI} = \frac{216.25}{EI}$$



$$\delta_{11} = \frac{2}{3}\times\frac{1}{2}\times\frac{1}{EI}\times 4 = \frac{4}{3EI}$$

$$\delta_{21} = \frac{1}{3}\times\frac{1}{2}\times\frac{1}{EI}\times 4 = \frac{2}{3EI}$$



$$\delta_{12} = \frac{1}{3}\times\frac{1}{2}\times\frac{1}{EI}\times 4 = \frac{2}{3EI}$$

$$\delta_{22} = \left(\frac{2}{3}\times\frac{1}{2}\times\frac{1}{EI}\times 4\right) + \left(\frac{2}{3}\times\frac{1}{2}\times\frac{1}{EI}\times 3\right)$$

$$= \frac{4}{3EI} + \frac{1}{EI}$$

$$= \frac{7}{3EI}$$

$$\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} M_A \\ M_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \Delta_{1L} \\ \Delta_{2L} \end{bmatrix}$$

$$(\beta)[R] + [\Delta] = 0$$

$$(\beta)(R) = -\Delta$$

$$\begin{bmatrix} 4/3EI & 2/3EI \\ 2/3EI & 7/3EI \end{bmatrix} \begin{bmatrix} M_A \\ M_B \end{bmatrix} = -\begin{bmatrix} 160/EI \\ 216.5/EI \end{bmatrix}$$

$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = \begin{bmatrix} \frac{4}{3EI} & \frac{2}{3EI} \\ \frac{2}{3EI} & \frac{7}{3EI} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{160}{EI} \\ -\frac{216.5}{EI} \end{bmatrix}$$

$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = \frac{1}{3EI} \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}^{-1} \frac{1}{EI} \begin{bmatrix} -160 \\ -216.5 \end{bmatrix}$$

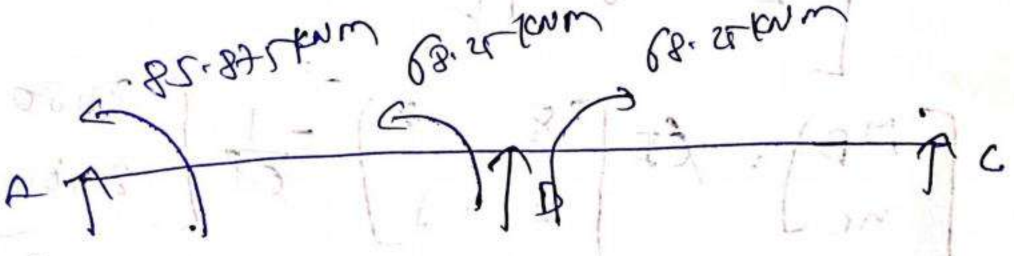
~~$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = 3EI$~~

$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = \left(\frac{1}{3EI}\right)^{-1} \frac{1}{28-4} \begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} -160 \\ -216.5 \end{bmatrix}$$

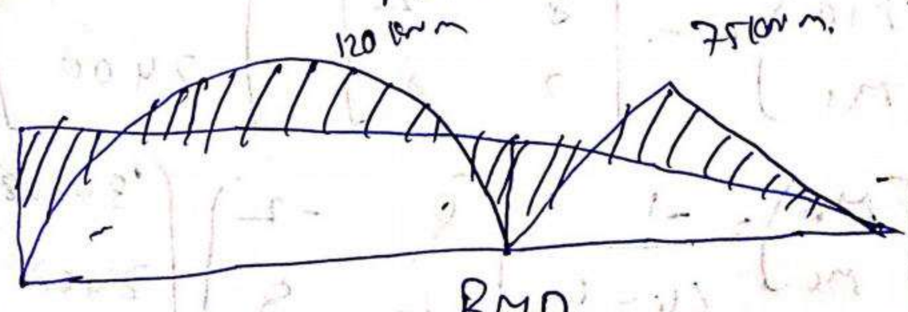
$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = \frac{3}{24} \begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}_{2\times2} \begin{bmatrix} -160 \\ -216.5 \end{bmatrix}_{2\times1}$$

$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -687 \\ -546 \end{bmatrix}_{2\times1}$$

$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = \begin{bmatrix} -85.875 \\ -68.25 \end{bmatrix}$$



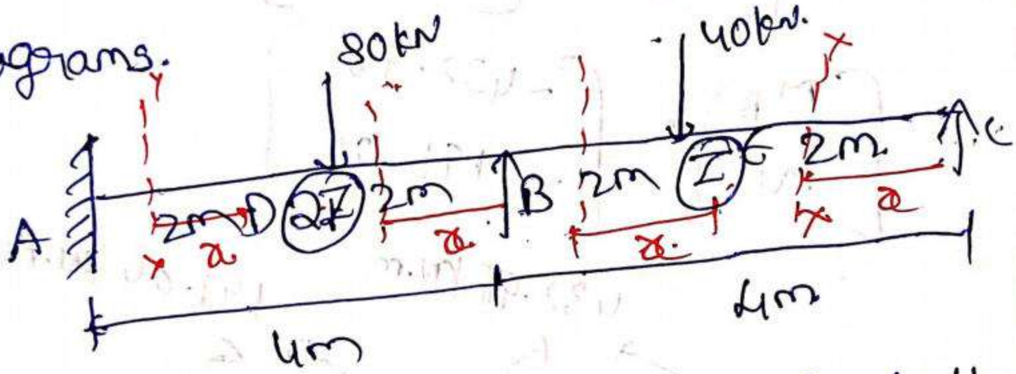
Final moments



BMD

Analysis of Continuous beam with sinking of supports by flexibility Matrix Method :-

Problem ② Analyse the continuous beam loaded as shown in figure using flexibility matrix method. The supports 'B' and 'C' settles by 10mm and 5mm respectively. Take $EI = 180 \times 10^{-11} N/mm^2$. Draw shear force and bending moment diagrams.



Solution:- Degree of static indeterminacy

$D_s = R - E - A$

$D_s = 5 - 3 - 0$

$\boxed{D_s = 2}$

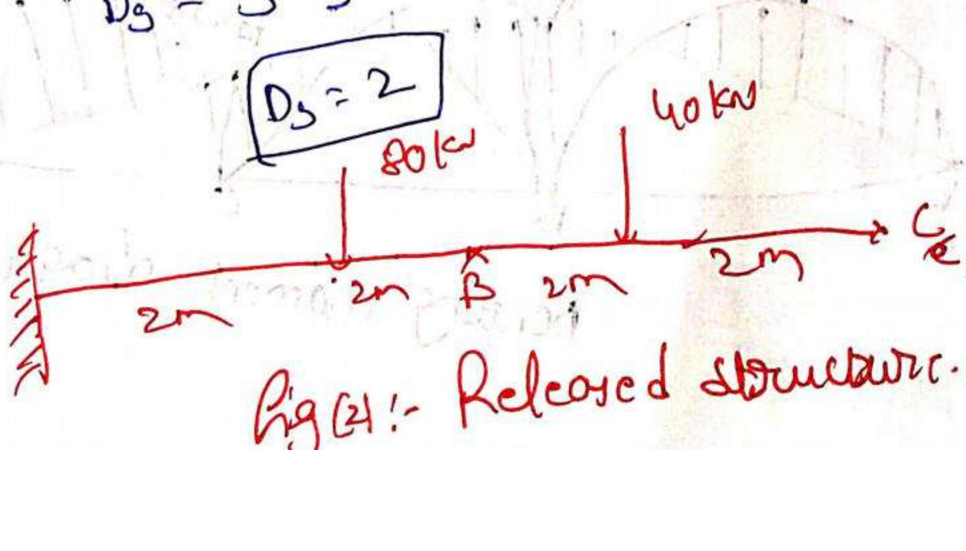


Fig (2) :- Released structure.

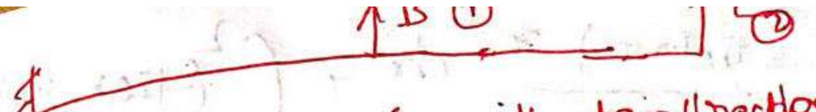


Fig (3) Co-ordinate directions.

$$\Delta = \int \frac{Mm\,dx}{EI}$$

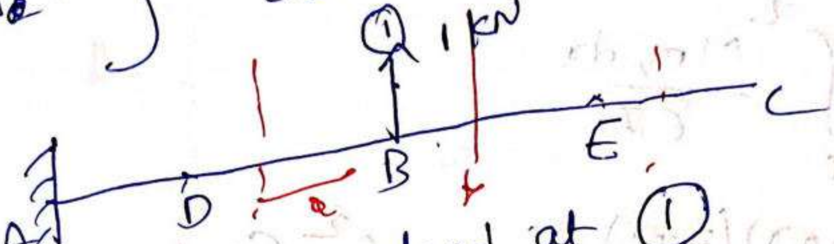


Fig (4) Unit load at ①

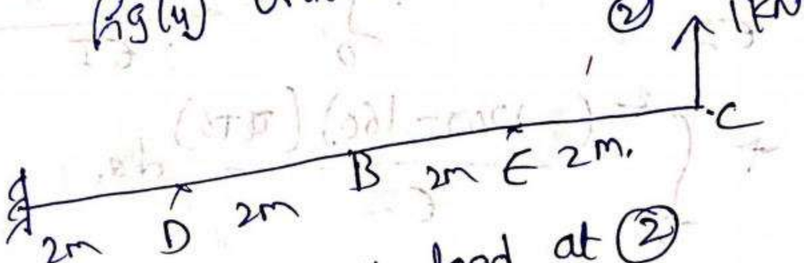


Fig (5) Unit load at ②

Segment	CE	EB	BD	DA
Origin	C	E	B	D
limits	0-2	0-2	0-2	0-2
EI	1	1	2	2
M	0	$-40x$	$-40(x+2)$	$-80x-40(x+4)$ $-120x-160$
m_1	0	0	x	$x+2$
m_2	x	$x+2$	$x+4$	$x+6$

$$\Delta = \int \frac{Mm\,dx}{EI}, \quad \Delta_{1L} = \int_0^L \frac{Mm_1\,dx}{EI}$$

$$\Delta_{2L} = \int^L \frac{Mm_2\,dx}{EI}$$

$$\Delta_{1L} = \int_0^2 \frac{-40(x+2)\,x}{EI}\,dx + \int_0^2 \frac{(-120x-160)(x+2)}{EI}\,dx$$

$$\Delta_{1L} = -1013.33/EI$$

$$\Delta_{2L} = \int_0^L \frac{M m_2\,dx}{EI}$$

$$\int_0^2 \frac{(-40x)(x+2)}{EI}\,dx + \int_0^2 \frac{(-40(x+2))(x+4)}{EI}\,dx$$

$$+ \int_0^2 \frac{(-120x-160)(x+6)}{EI}\,dx$$

$$\Delta_{2L} = \frac{-2880}{EI}$$

$$\delta_{11} = \int_0^L \frac{m_1 m_1\,dx}{EI} = \frac{10.67}{EI}$$

$$\delta_{12} = \delta_{21} = \int_0^L \frac{m_1 m_2\,dx}{EI} = \frac{26.6}{EI}$$

$$\delta_{22} = \int_0^L \frac{m_2 m_2}{EI}\,dx = \frac{96}{EI}$$

$[D] - [D_L] = [\delta][P]$ $\quad P = KP$

$\Delta_1 = -10mm = -0.01m$ $\quad (D - D_L) = [\delta][P]$

$\Delta_2 = \Delta_C = 5mm = -0.005m$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} - \begin{bmatrix} \Delta_{1L} \\ \Delta_{2L} \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$\begin{bmatrix} -0.01m \\ -0.005 \end{bmatrix} - \begin{bmatrix} -1013.33/EI \\ -\frac{2880}{EI} \end{bmatrix} = \begin{bmatrix} 10.67/EI & \frac{26.67}{EI} \\ \frac{26.67}{EI} & \frac{96}{EI} \end{bmatrix} \begin{bmatrix} R_B \\ R_C \end{bmatrix}$$

$EI = 18000$

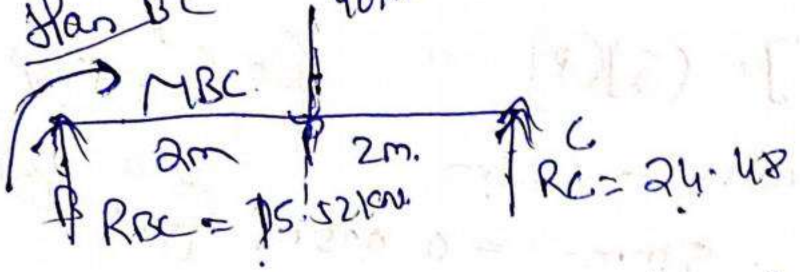
LHS

$$\begin{bmatrix} -0.01 \\ -0.005 \end{bmatrix} - \begin{bmatrix} -1013.33/18000 \\ -2880/18000 \end{bmatrix} = \begin{bmatrix} 0.046 \\ 0.155 \end{bmatrix}$$

$$\begin{bmatrix} 0.046 \\ 0.155 \end{bmatrix} = \begin{bmatrix} 10.67/18000 & \frac{26.67}{18000} \\ \frac{26.67}{18000} & \frac{96}{18000} \end{bmatrix} \begin{bmatrix} R_B \\ R_C \end{bmatrix}$$

$$\begin{bmatrix} R_B \\ R_C \end{bmatrix} = \begin{bmatrix} 16.71 \text{ kN} \\ 24.48 \text{ kN} \end{bmatrix}$$

Span BC



$\Sigma F_y = 0 \quad R_{BC} + R_C - 40 = 0$

$R_{BC} + 24.48 - 40 = 0$

$\underline{R_{BC} = 15.52 kN}$

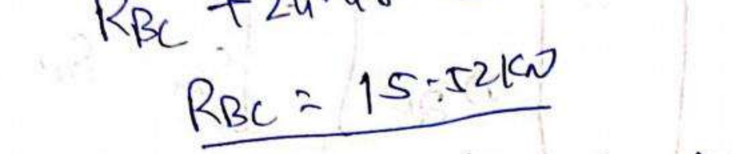
$\Sigma M_B = 0 \quad -R_C \times 4 + (40 \times 2) + M_{BC} = 0$

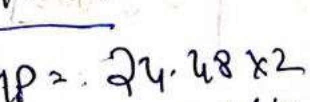
$-24.48 \times 4 + 80 + M_{BC} = 0$

$\underline{M_{BC} = 17.92 kNm}$

$M_D = 24.48 \times 2$
$= 48.96 kNm$

Span AB





$\underline{R_{BA} = 1.19 kN}$

$M_O = 1.19 \times 2 + ...$
$M_O = 20.31 kN...$

$R_B = R_{BA} + R_{BC}$

$16.71 = R_{BA} + 15.52$

$R_{BA} = 16.71 - 15.52$

$\boxed{R_{BA} = 1.19 kN}$

$M_B = M_{BA} + M_{BC} = 0$

~~MD = 1.19×2+~~

M_{BC}

$M_{AB} = 80 \times 2 - 17.92 - R_{BA} \times 4 + M_{AB} = 0$

$M_{AB} = -137.69 \text{ kNm}$

$\Sigma F_y = 0 \quad R_A + R_{BA} - 80 = 0$

$R_A = 78.015 \text{ kN}$

$R_A = 78.015 \text{ kN}$

$R_{BA} = 1.19 \text{ kN}$

$R_{BC} = 15.52 \text{ kN}$

$R_C = 24.48 \text{ kN}$

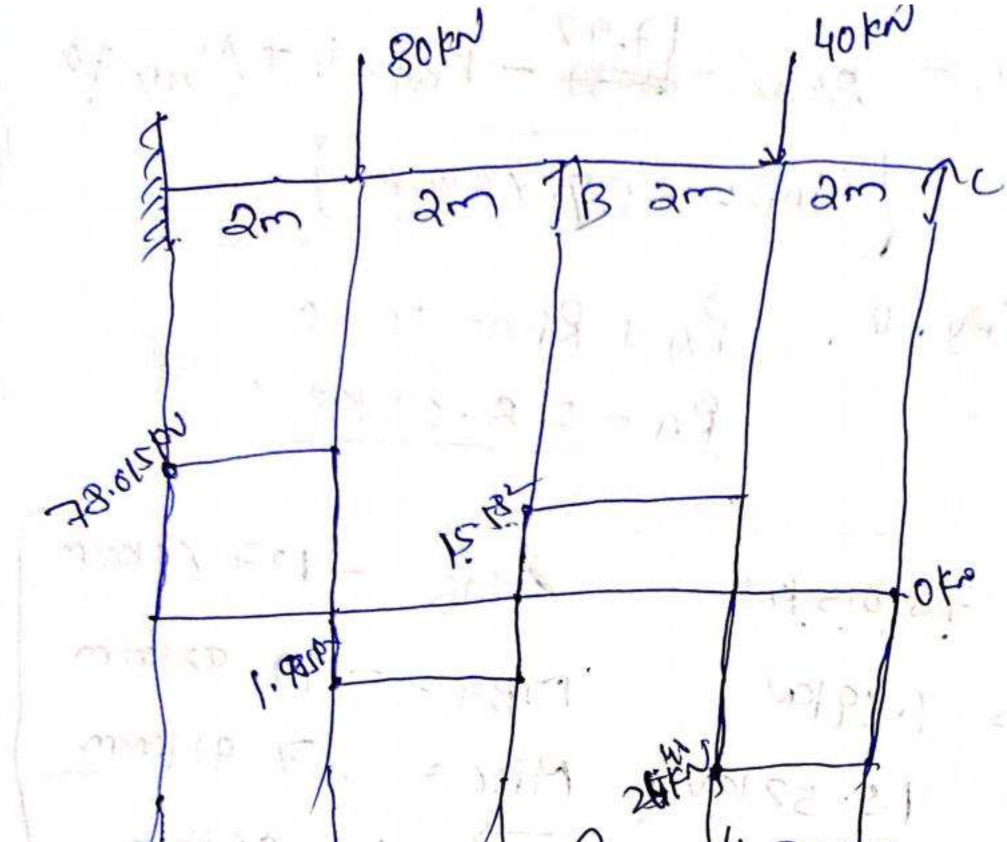
$M_{AB} = -137.69 \text{ kNm}$

$M_{BA} = -17.92 \text{ kNm}$

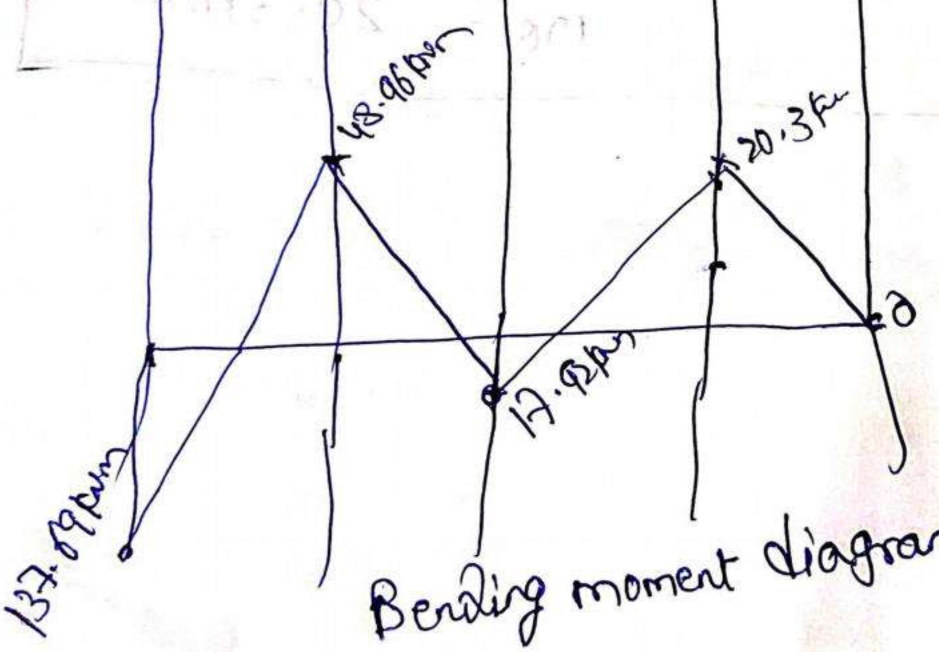
$M_{BC} = 17.92 \text{ kNm}$

$\overrightarrow{MP} = 48.96 \text{ kNm}$

$M_\theta = 20.3 \text{ kNm}$



Shear force diagram



Bending moment diagram.

Stiffness Matrix Method

29/1/2021

Step ①:- we Calculate Degree of Freedom (DOF).

Step ②:- Find Fixed end moment (FEM).

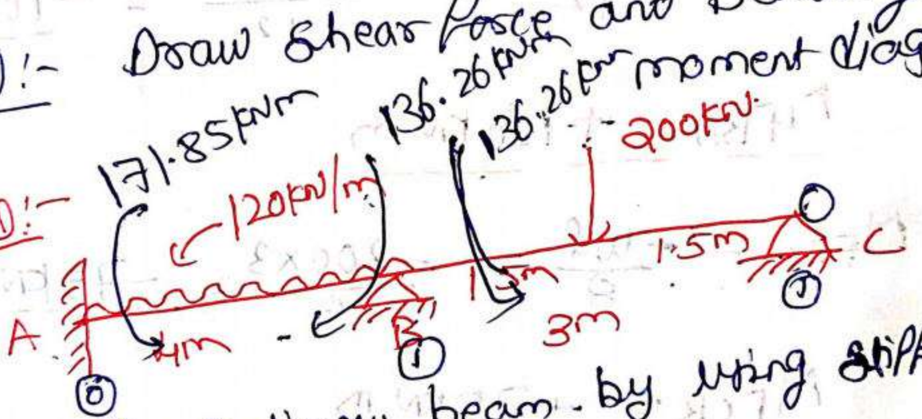
Step ③:- Stiffness Matrix (K).

Step ④:- Calculate Unknown displacements

Step ⑤:- Final end moments.

Step ⑥:- Draw Shear force and Bending moment diagram

Problem ①:-



Analyse the continuous beam by using stiffness matrix Method.

Step ①:- Degree of freedom (DOF):-

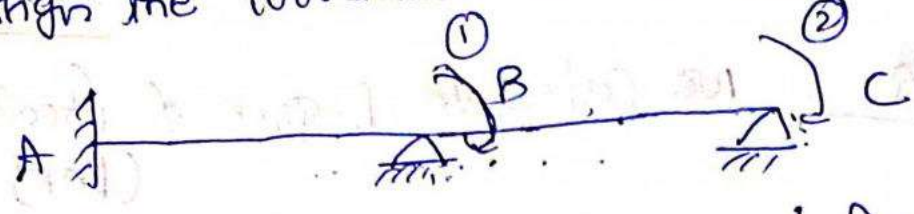
DOF = 0 + 1 + 1 = 2 (or) 3J − R

= 3(3) − 7

= 9 − 7 = 2

DOF = 2

Assign the coordinate.



$\therefore$ Displacement Matrix $[D] = \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}$

$\therefore [D] = \begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix}$

$P = KD$

Step②:- Fixed end moment (FEM)

$M_{FAB} = -\frac{wl^2}{12} = -\frac{120 \times 4^2}{12} = -160 \text{ KN·m}$

$M_{FBA} = +160 \text{ KN·m}$

$M_{FBC} = -\frac{wl}{8} = \frac{-200 \times 3}{8} = -75 \text{ KNm}$

$M_{FCB} = +75 \text{ KN·m}$

WKT $[P] = [P_i - P_L]$

$\begin{bmatrix} P_{1i} \\ P_{2i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We need to find P_{1L} & P_{2L}

$P_{1L} = M_{FBA} + M_{FBC}$

$F_{2L} = M_{FCB} = +75 \text{ kNm}$

$\therefore$ Force Matrix $[P] = \begin{Bmatrix} P_{1i} - P_{1L} \\ P_{2i} - P_{2L} \end{Bmatrix}$

$$(P) = \begin{bmatrix} 0 - 85 \\ 0 - 75 \end{bmatrix}$$

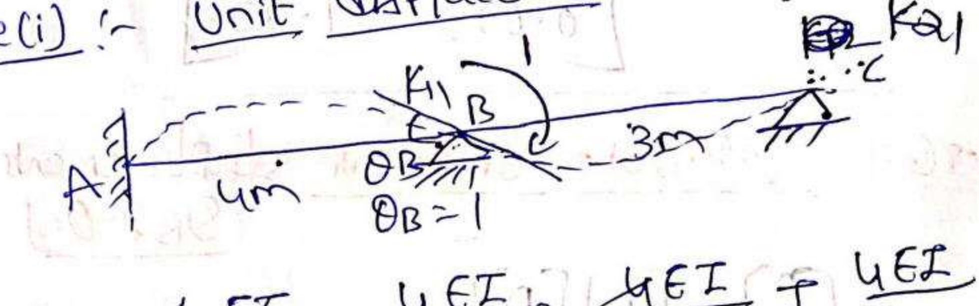
$$[P] = \begin{bmatrix} -85 \\ -75 \end{bmatrix} \text{ kNm}$$

Step ③:- Stiffness matrix (K):-

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

K_{11}, K_{12}, K_{21} & K_{22} are the stiffness coefficients

Case (i):- Unit displacement at coordinate ①



$$K_{11} = \frac{4EI}{L} + \frac{4EI}{L} = \frac{4EI}{4} + \frac{4EI}{3}$$

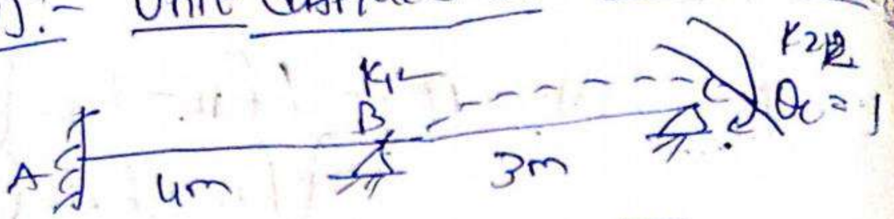
$$K_{11} = EI + \frac{4}{3}EI$$

$$\boxed{K_{11} = 2.33EI}$$

$K_{21} = \frac{2EI}{L} = \frac{2EI}{3} = 0.67EI$

$\boxed{K_{21} = 0.67EI}$

Case(ii):- Unit displacement at coordinate (2)



$$K_{12} = \frac{2EI}{L} = \frac{2EI}{3} = 0.67EI$$

$$\boxed{K_{12} = 0.67EI}$$

$$K_{22} = \frac{4EI}{L} = \frac{4EI}{3} = 1.33EI$$

$$\boxed{K_{22} = 1.33EI}$$

$\therefore$ Stiffness matrix $[K] = \begin{bmatrix} 2.33EI & 0.67EI \\ 0.67EI & 1.33EI \end{bmatrix}$

$$(K) = EI\begin{bmatrix} 2.33 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}$$

Step (4):- Calculate Unknown displacement $(\theta_B \,\&\, \theta_C)$

$$[P] = [K][D]$$

$$\begin{bmatrix} -85 \\ -75 \end{bmatrix} = EI\begin{bmatrix} 2.33 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

$$\begin{bmatrix} \theta_B \end{bmatrix} = \frac{1}{\;} \begin{bmatrix} 2.33 & 0.67 \end{bmatrix}^{-1}\begin{bmatrix} -85 \\ -75 \end{bmatrix}$$

$$\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} -23.75 \\ -44.34 \end{bmatrix}$$

Step (5) :- Final end moments :-

$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

($\theta_A \to 0$, $\frac{3\Delta}{L} \to 0$)

$$M_{AB} = -160 + \frac{2EI}{4}\left(\frac{-23.75}{EI}\right)$$

$$\boxed{M_{AB} = -171.85 \text{ kNm}}$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

($\theta_A \to 0$, $\frac{3\Delta}{L} \to 0$)

$$M_{BA} = 160 + \frac{2EI}{4}\left(2\left(\frac{-23.75}{EI}\right)\right)$$

$$\boxed{M_{BA} = 136.26 \text{ kNm}}$$

$$M_{BC} = M_{FBC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

($\frac{3\Delta}{L} \to 0$)

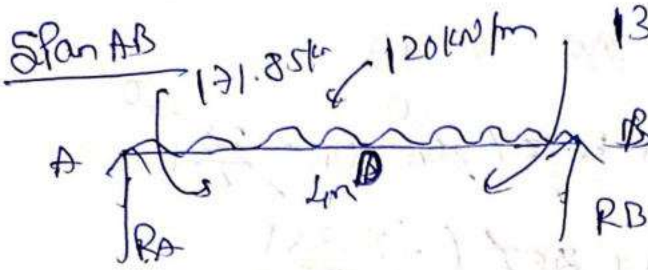
$$M_{BC} = -75 + \frac{2EI}{3}\left[2\left(\frac{-23.75}{EI}\right) + \left(\frac{-44.34}{EI}\right)\right]$$

$$\boxed{M_{BC} = -136.26 \text{ kNm}}$$

$$\boxed{M_{CB} = 0}$$

Step (8) :- Draw Shear force and Bending moment diagram

Span AB



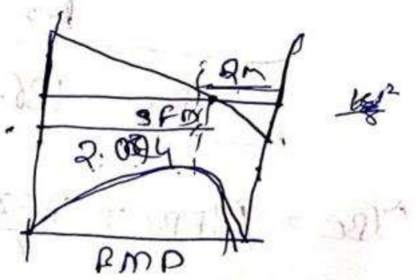
$\Sigma M_B = 0$ $\quad R_A \times 4 - 120 \times 4 \times 2 - 171.85 + 136.26$

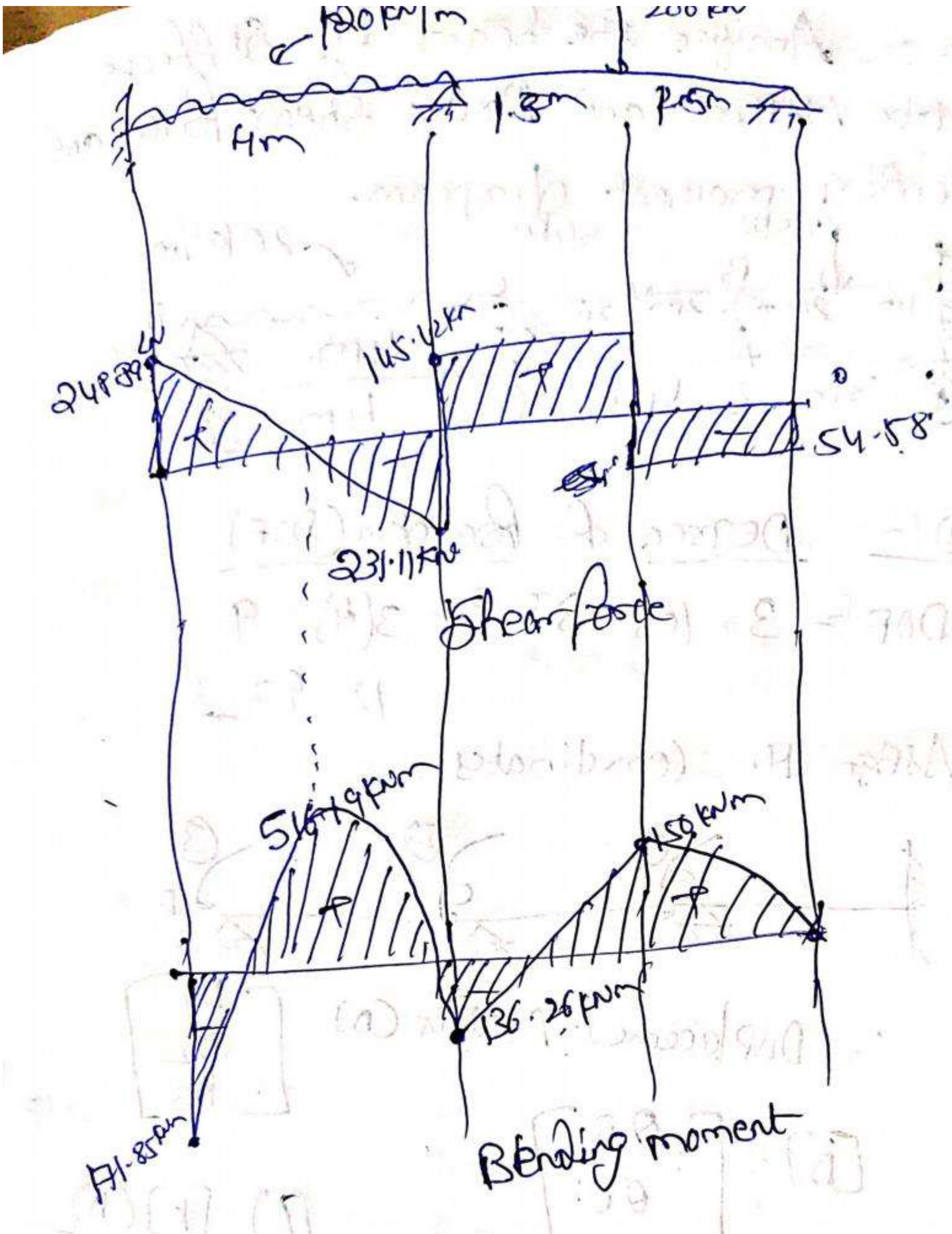
$R_A = 248.89$ kN

$\Sigma F_y = 0$ $\quad R_A + R_{BA} - 120 \times 4 = 0$

$R_{BA} = 231.11$ kN

$\overrightarrow{M_D} =$

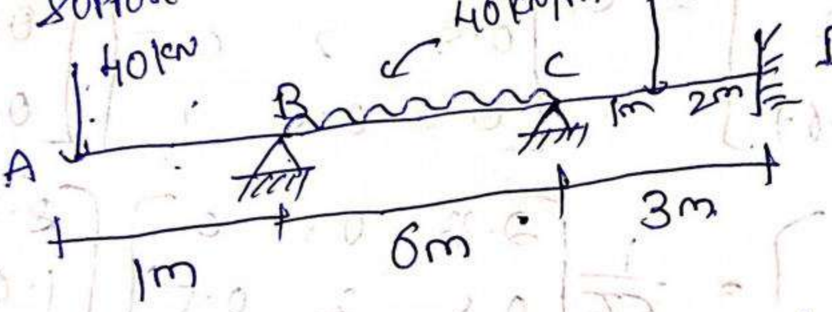



120 kN/m
4m
145.42 kN
54.58
231.11 kN
Shear force
150 kNm
136.26 kNm
Bending moment

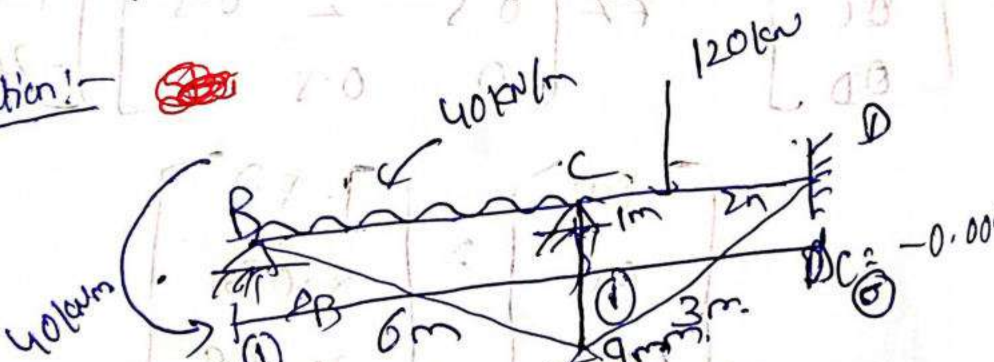
1/2/2021

Analysis of Continous beam with Sinking of Support by using Stiffness Matrix Method.

Problem:- Analyse the beam shown in figure by using Stiffness matrix method and Draw Shear force and bending moment diagrams. If support 'C' sinks by 9mm. Take EI = 1000 kNm²



Solution:-



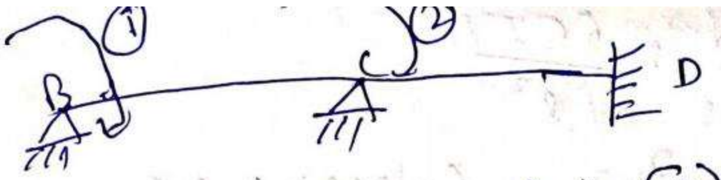
Step①:- Degree of Freedom (DOF)

DOF = 1+1+0 = 2

(or)

DOF = 3J − R = 3(3) − 7

= 9 − 7 = 2



$\therefore$ Displacement matrix $(D) = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$

$$[D] = \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

Step(2):- Force Matrix

Fixed End moment (FEM):-

$$M_{FBC} = \frac{-wl^2}{12} - \frac{6EI\Delta}{l^2}$$

$$M_{FBC} = \frac{-40 \times 6^2}{12} - \frac{6 \times 1000 \times 0.009}{6^2}$$

$$M_{FBC} = -120 \, kNm$$

$$M_{FCB} = \frac{+wl^2}{12} - \frac{6EI\Delta}{l^2}$$

$$M_{FCB} = \frac{40 \times 6^2}{12} - \frac{6 \times 1000 \times 0.09}{6^2}$$

$$M_{FCB} = 118.5 \, kNm$$

$$M_{FCD} = \frac{-Wab^2}{l^2} + \frac{6EI\Delta}{l^2}$$

$$M_{FCD} = \frac{-120 \times 1 \times 2^2}{2^2} + \frac{6 \times 1000 \times 0.004}{2^2}$$

$$M_{FDC} = \frac{wa^2b}{l^2} + \frac{6EI\Delta}{l^2}$$

$$M_{FDC} = \frac{120 \times 1^2 \times 2}{3^2} + \frac{6 \times 1000 \times 0.009}{3^2}$$

$$M_{FDC} = 32.67\ kNm$$

$\therefore$ Force Matrix $(P) = [P_i - P_L]$

$$\begin{bmatrix} P_{1i} \\ P_{2i} \end{bmatrix} = \begin{bmatrix} -40 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P_{1L} \\ P_{2L} \end{bmatrix} = \begin{bmatrix} -121.5 \\ 71.7 \end{bmatrix}$$

$P_{1L} = M_{FBC} = -121.5\ kNm$

$P_{2L} = M_{FCB} + M_{FCD} = 118.5 - 47.33$

$$P_{2L} = 71.17\ kNm$$

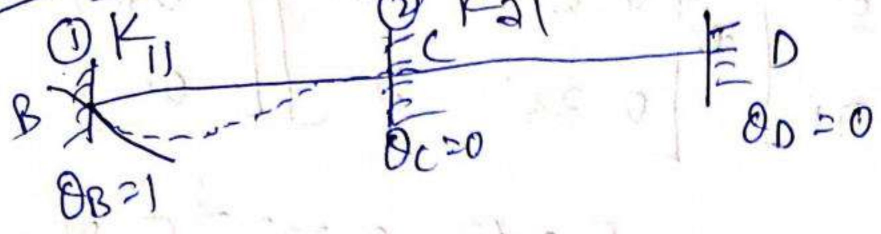
Force Matrix $(P) = \begin{bmatrix} P_{1i} - P_{1L} \\ P_{2i} - P_{2L} \end{bmatrix}$

$$[P] = \begin{bmatrix} -40 + 121.5 \\ 0 - 71.71 \end{bmatrix}$$

$$[P] = \begin{bmatrix} 81.5 \\ -71.71 \end{bmatrix} kNm$$

Step ③ :- Stiffness Matrix (K) :-

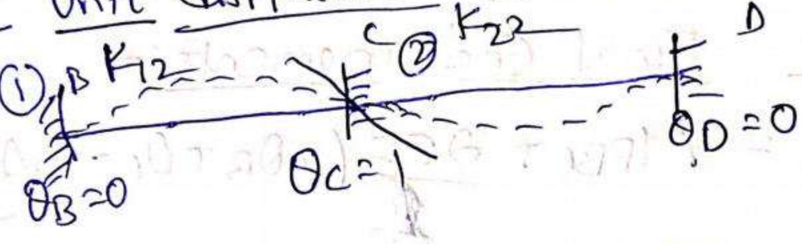
Case (i) :- Unit displacement at 1st Coordinate



$$K_{11} = \frac{4EI}{L} = \frac{4EI}{6} = 0.67EI$$

$$K_{21} = \frac{2EI}{L} = \frac{2EI}{6} = 0.33EI$$

Case (ii) :- Unit displacement at 2nd Coordinate



$$K_{12} = \frac{2EI}{L} = \frac{2EI}{6} = 0.33EI$$

$$K_{22} = \frac{4EI}{L} + \frac{4EI}{L} = \frac{4EI}{6} + \frac{4EI}{3}$$

$$K_{22} = 2EI$$

$\therefore$ Stiffness Matrix (K) = $\begin{bmatrix} 0.67EI & 0.33EI \\ 0.33EI & 2EI \end{bmatrix}$

$$K = EI \begin{bmatrix} 0.67 & 0.33 \\ 0.33 & 2 \end{bmatrix}$$

Step (4) :- Calculate unknown displacements (θs)

We know that $[P] = [K][D]$

$$\begin{bmatrix} 81.5 \\ -71.71 \end{bmatrix} = EI \begin{bmatrix} 0.67 & 0.33 \\ 0.33 & 2 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

$$\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 0.67 & 0.33 \\ 0.33 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 81.5 \\ -71.7 \end{bmatrix}$$

$$\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 151.75/EI \\ -60.91/EI \end{bmatrix}$$

Step (5) :- Final End moments :-

$$M_{BC} = M_{FBC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$M_{BC} = -121.5 + \frac{2EI}{6}\left(2 \times \frac{151.75}{EI} - \frac{60.91}{EI} - \frac{3 \times 0.009}{6}\right)$$

~~$M_{BC} = 212$.~~

$$M_{BC} = -121.5 + \frac{2 \times 1000}{6}\left(\frac{303.5}{1000} - \frac{60.91}{1000} - 0.0045\right)$$

$$M_{BC} = -121.5 + 333.33\,(0.23)$$

$$M_{BC} = -121.5 + 79.33$$

$M_{BC} = -1.7\ 13\ kN\,m$