UNIT –I

DEFLECTION OF DETERMINATE BEAMS

Objective:

To familiarize with the deflection of simple determinate beams.

Syllabus:

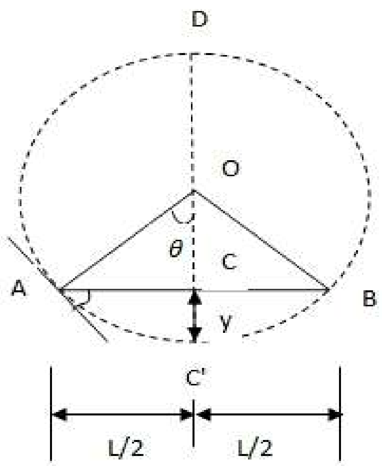
Deflection and slope of a beam subjected to uniform bending moment relation between slope, deflection and radius of curvature & Differential equation for the elastic line of a loaded beam. Determination of slope and deflection for cantilever, Simply Supported beam and over hanging beams subjected to point loads and UDL by Macaulay's and Moment area methods. types of springs in series and parallel-deflection of closely coiled helical springs under axial pull only.

Learning Outcomes:

Student will be able to

- Determine the slope and deflection for determinate beams using Macaulay's method.
- Determine the slope and deflection for determinate beams using Moment area method.

- **Deflection and slope of a beam subjected to uniform bending moment:**



A beam AB of length L is subjected to uniform bending moment M.

The initial position of the beam is shown by ACB , where as the deflected position is shown by AC^1 B.

Let R= radius of curvature of the deflected beam

Y= deflection of the beam at the centre

I= Moment of inertia of the beam

= slope of the beam at the end A

= where is in radius

As is slope is

Now AC=CB=

From the geometry of circle,

The deflection y is a small quantity. Hence the square of a small quantity will be negligible.

Bending equation =

is the central deflection of a beam which bends in a circular arc.

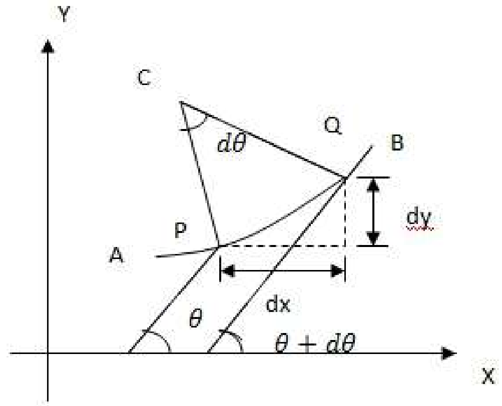
Slope:

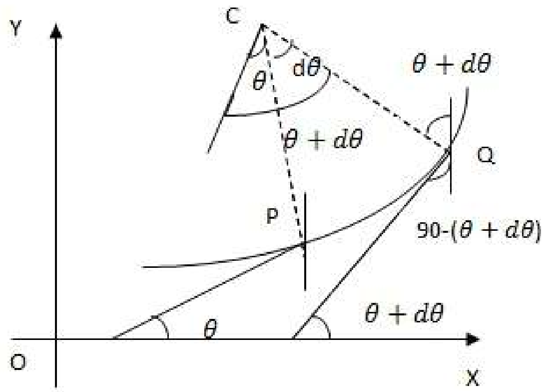
From triangle AOB, sin = =

is very small, sin=

Relation between slope, deflection and radius of curvature:

Let the curve AB represents the deflection of a beam as shown in fig.





Consider a small portion PQ of this beam. Let the tangents at P and Q make angle and with x-axis. Normal at P and Q will meet at C such that PC= QC= R. The point C is known as centre of curvature of the curve PQ.

Let the length of PQ is equal to ds.

From the geometry of fig

R =ds

Differentiate the above equation w.r.t x, we get

M= E.

M = EI

Differentiate the above equation w.r.t 's'

If the curvature is very small then is also small and its square is negligible.

Moment M=EI

Shear force F= = EI

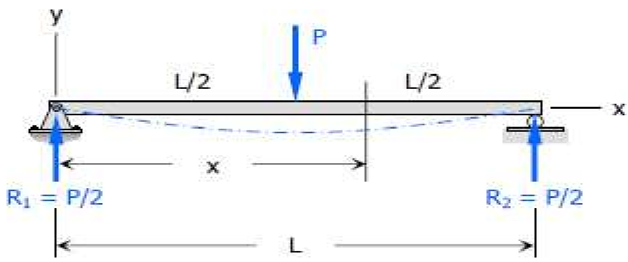
Rate of loading -w=

- Various Methods determining Slope and deflection at a section in a loaded beam
    1. Double integration method
    2. Moment area method

3. Macaulay's method

Double integration method:

Maximum deflection δ in a simply supported beam of length L carrying a concentrated load P at midspan.



EI y''= Px−P ⟨x−L⟩

EI y'=Px^2−P $\langle x-L\rangle^2$+ C_1

EI y'=Px^3−P $\langle x-L\rangle^3$+ C_1x+C_2

At x = 0, y = 0, therefore, $C_2 = 0$

At x = L, y = 0

0=PL^3−P $\langle L-L\rangle^3$+ C_1L

0=PL^3-PL^3+ C_1L

C1=−PL^2

Thus,

EI y=Px^3− P $\langle x-L\rangle^3$−PL^2x

Maximum deflection will occur at x = ½ L (midspan)

=P −P− P( )

= P−0−P

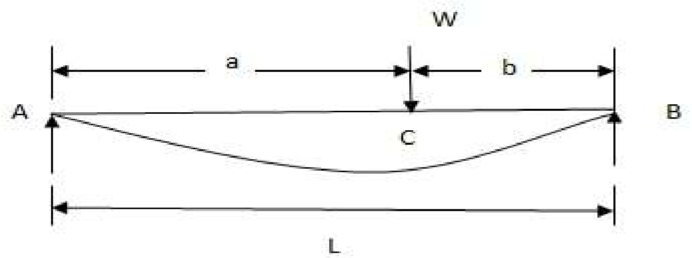
=

Therefore

Macaulay's method:

This is a convenient method for determining the slope and deflections of the beam subjected to point loads.

Maximum deflection δ in a simply supported beam of length L carrying a eccentric point load P at free end:



The bending moment at any section between A and C at a distance x from A is given by

The bending moment at any section between C and B at a distance x from A is given by

The B.M for all the sections of the beam is expressed in a single equation

we know that

M=EI

EI

integrating the above equation

EI

integrating the above equation

EI

Apply boundary conditions

At x=0, y=0 and

At x=L, y=0

At x=0 and y=0, then

At x=L, y=0, then)

substitute the values of , in the above equation

EI

EI

Slope is maximum at A or B

at A =

At x=0,

EI

The deflection under the load is , substitute x=a, then we get

EI

Maximum deflection :

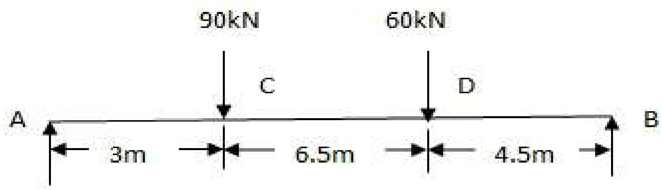
The slope is zero at the point of maximum deflection,

x=

for put the value of x in deflection equation

EI

Example: Find the deflection of the girder at the points under the loads. and also find the maximum deflection. Take $I=64x\ 10^{-4}m^4$ and $E= 2.1x10^5N/mm^2$.



EI

Integrating on both sides

EI

Again integrating on both sides

EI

At x=0, y=0 then

At x=14, y=0 then

Deflection at C, substitute x=3

EI

= -2.93mm

Deflection at D, substitute x=9.5m

= -3.73mm

Maximum deflection

Let us assume that the deflection will be maximum at section between C and D. The slope equation at the section is equal to zero at the maximum deflection.

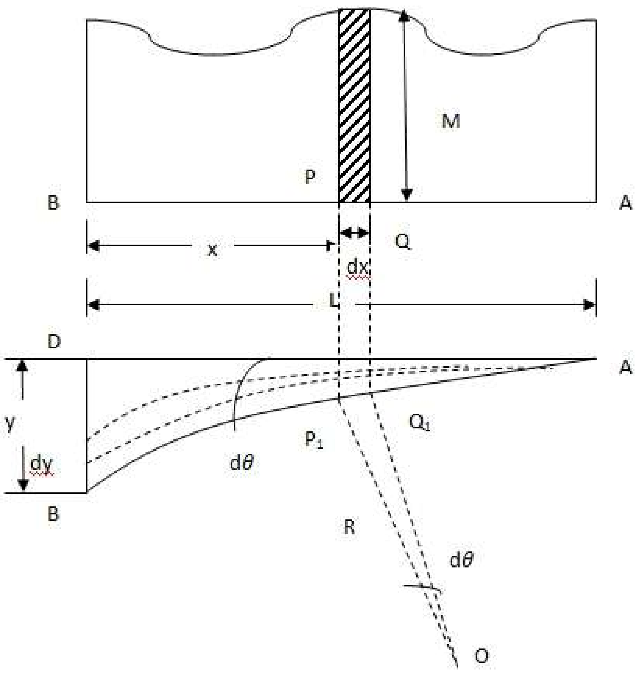
EI

x=6.87m

EI

Moment Area method:

This method is convenient in case of beam act upon with point loads in which case bending moment area consist of triangles and rectangular. This method is mainly explained by Mohr's theorems.



Mohr's theorem 1:

It states that the change of slope between any two points on an elastic curve is equal to area of bending moment diagram between these points divided by flexural rigidity (EI).

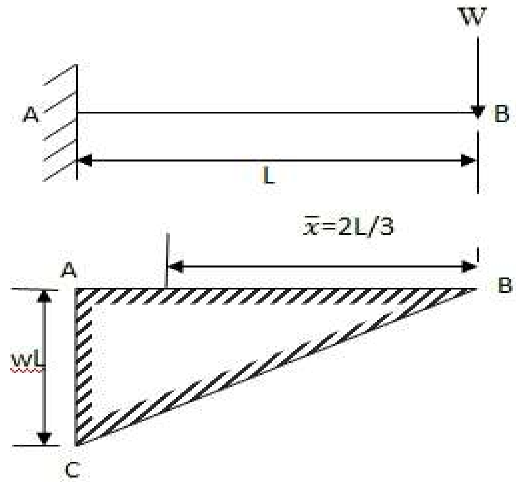
Mohr's theorem 2:

It states that the intercept taken on a vertical reference line of tangents at any two points on an elastic curve is equal to the moment of the bending moment diagram between these points above the line divided by flexural rigidity.

Deflection and slope of a cantilever by Moment area method:

Cantilever carrying a point load at the free end:

The fig shows a cantilever of length L fixed at end A and free at the end B.It carries a point load W at B.



At the fixed end A, the slope and deflection are zero.

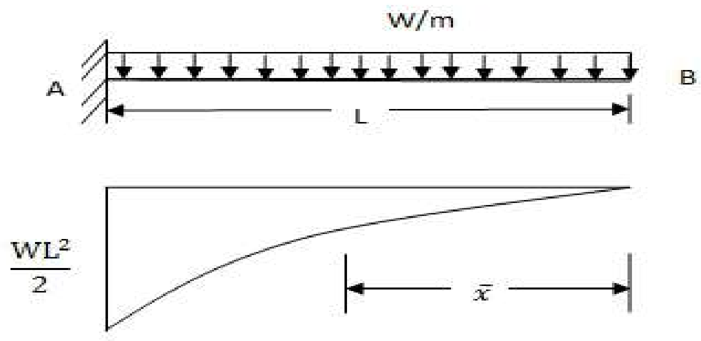
Then according to moment area method,

And

A= Area of B.M diagram between A and B =
= Distance of C.G of area of B.M diagram from B=

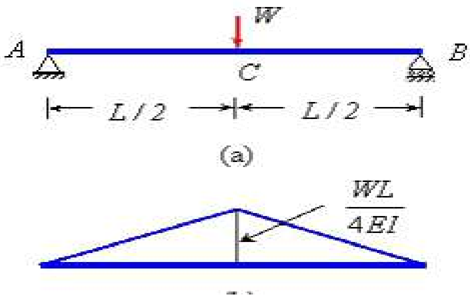
Cantilever carrying a uniformly distributed load:

The fig shows a cantilever of length L fixed at end A and free at the end B. It carries a uniformly distributed load of w/unit length over the entire length.



Determine the end slope and deflection of the simply supported beam carrying a point

load at the centre:



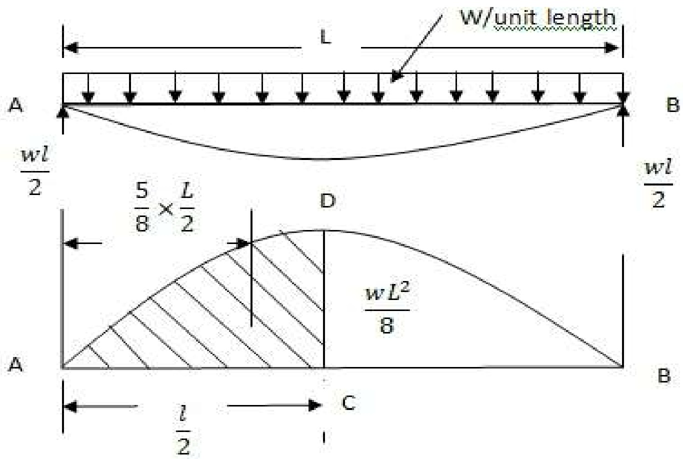
Slope at A

But area of B.M diagram between A and C = Area of triangle ACD

=

=

Determine the end slope and deflection of the simply supported beam carrying a uniformly distributed load at the centre:



But area of B.M diagram between A and C = Area of parabola ACD

=

=

Example: Find the slope and deflection of cantilever beam at the free end using moment area method.

to find the area of the B.M diagram, divide the fig into two triangles and one rectangle.

area A_1=

area A_2==

area A_3=

Total area of B.M diagram,

A= A_1+ A_2+ A_3=

=0.005 radians

++ =

C

= $7x10^{13}Nmm^3$

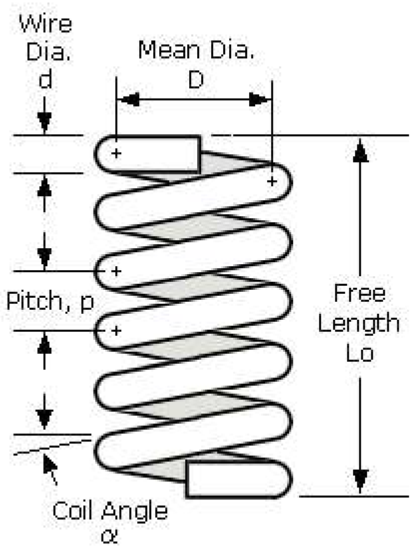
Close-Coiled Helical Spring

Axial Load:

a) *Neglecting curvature and direct shear effects:*

Consider a Closely coiled helical spring as shown in fig. under the action of axial load.

Let,



W = Axial Load

D = Mean coil diameter

d = Dia. of Spring wire

δ = Axial deflection

G = Modulus of rigidity

Θ = Angular Deflection

n = No. of active coils

= Maximum shearing stress induced

The following assumptions are made:

i. An element of an axially loaded helical spring behaves essentially as a straight bar in pure torsion.
ii. The planes perpendicular to the bar axis do not wrap or distort during deformation. As a result of this the shearing stress will have a linear distribution along the radius.

Fig. Shows the circular sectional element of the spring wire under torsion. Torque on the spring acting about the axis of the spring.

At any radius x from the center 'O' of the wire, the shearing stress is,

The torque dT taken up by a ring of width dr at a radius r will be,

Total torque T =

------ (1)

Calculate rotation and deflection of the spring, consider the elementary angle dΘ through which one cross section rotates w.r.t other.

Θ =

Θ = ------ (2)

From eq (1) & (2)

Now, =

& Θ =

Now, deflection δ =

Θ =

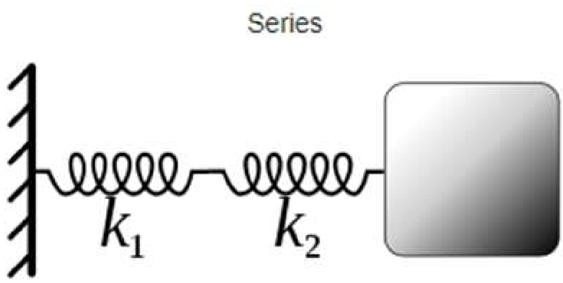
Stiffness, K = =

Direct Shear Stress,

Therefore, Maximum resultant shear stress = +

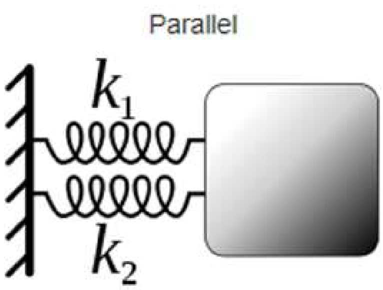
Springs in Series: when two springs of different stiffness are joined end to end to carry a common load w, they are said to be connected in series, as shown in fig

Total deflection,



Where K is the combined stiffness

Springs in parallel: when two springs are joined in such a way that they have a common deflection, they are said to be connected in parallel two different.



W=W1+W2

UNIT-1

Assignment-Cum-Tutorial Questions

A. Questions testing the remembering / understanding level of students

I) Objective Questions

1. Slope at a point in a beam is the
   (a) Vertical displacement (b) Angular displacement (c) Horizontal displacement (d) None
2. Deflection at a point in a beam is the
   (a) Vertical displacement (b) Angular displacement (c) Horizontal displacement (d) None
3. Maximum deflection in a S.S. beam with W at centre will be
   (a) $WL^3/36EI$ (b) $WL^3/24EI$ (c) $WL^3/48EI$ (d) $WL^3/96EI$
4. Maximum slope in a S.S. beam with W at center will be
   (a) At the supports (b) At the center
   (c) In between the support and the center (d) None
5. Maximum deflection in a cantilever beam with UDL 'w' over the entire span will be
   (a) At the left hand support (b) At the Right support (c) At the center (d) None
6. Deflection under the load in a S.S.beam with 'W' not at the center will be
   (a) $4Wa^2b^2/3EIL$ (b) $2Wa^2b^2/3EIL$ (c) $Wa^2b^2/3EIL$ (d) None
7. Distance of maximum deflection from the center in a S.S.Beam with 'W' not at the center will be

(a) $[2(L^2 — b^2)/3]^{0.5}$ (b) $[(L^2 — b^2)/3]^{0.5}$

(c) $[(3L^2 — b^2)/3]^{0.5}$ (d) $[4(L^2 — b^2)/3]^{0.5}$

8. Difference in slopes between two points A and B by the moment area method is given by

   (a) Area of BMD between A and B/2EI (b) Area of BMD between A and B/3EI

   (c) Area of BMD between A and B/EI (d) Area of BMD between A and B/4EI

9. Difference in deflections between two points A and B by the moment area method is given by

   (a) (Area of BMD between A and B). /2EI

   (b) (Area of BMD between A and B). /3EI

   (c) (Area of BMD between A and B). /EI

   (d) None

10. Macaulay's method is more convenient for beams carrying
    (a) Multi concentrated loads (b) Multi number of UDL
    (c) Multi-concentrated and multi UDL loads (d) None

11. The ratio of maximum deflections of a cantilever beam of span L with (i) a load W at free end (ii) a U.D.L over entire length of total W is given by
    (a) 3/8 (b) 8/3 (c) 5/8 (d) 8/5

12. If the depth of a cantilever is doubled and width is halved, the deflection of a cantilever due to a point load at free end changes in the ratio.
    (a) 1/2 (b)1/4 (c) 1/8 (d) 1/16

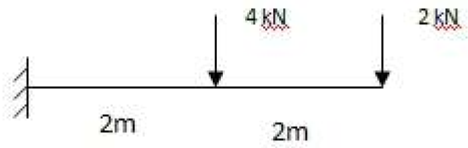
II) *Problems:*

1. A beam of length 4.8 m and of uniform rectangular section is simply supported at its ends. It carries a uniformly distributed load of 10 kN/m run over the entire length. Calculate the width and depth of the beam if permissible bending stress is 7 N/mm^2 and maximum deflection is not to exceed 0.95 cm. Take E for beam material = 1.05 x 10^4 N/mm^2.
2. A simply supported beam of 4 m span carries a UDL of 20 kN/m on the whole span and in addition carries a point load of 40 kN at the centre of span. Calculate the slope at the ends and the maximum deflection of the beam. Take E = 200 GN/m^2 and I = 5000 cm^4.
3. A cantilever 3 m long is of rectangular section 120 mm wide and 240 mm deep it carries a UDL of 2.5 kN per meter length for a length of 1.5 m from the fixed end and a point load of 1 kN at the free end. Find the deflection at the free end. Take E = 10 GN/ m^2.
4. A cantilever 3m long carries two point loads, 60 kN each, at distance of 0.75 m and 1.75 m respectively from the fixed end. Determine the deflection at the free end. Take E= 200GN/m^2 and I = 12689400 cm^4.
5. A steel girder of uniform section, 14 meters long, is simply supported at its ends. It carries concentrated loads of 120 kN and 80 kN at two points 3 meters and 4.5meters from the two ends respectively. (a) Calculate the deflection of the girder at the two

points under the two loads.(b) The maximum deflection. Use Macaulay's Method. Take: $I = 16 \times 10^4\ m^4$, and $E = 210 \times 10^6\ KN/m^2$.

6. A simply supported beam of span 10m is loaded with a UDL of 5000 N/m over a length of 3 m from the left end. Find the maximum deflection of the beam. Take E = 0.2 MN/mm^2 and I= 3000 cm^4.

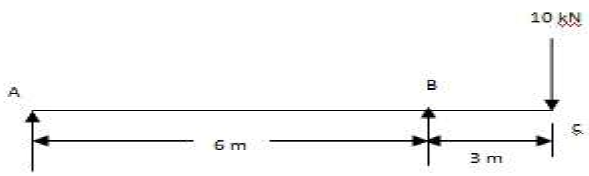
7. The cantilever beam shown in Fig. has a rectangular cross-section 50 mm wide by h mm high. Find the height h if the maximum deflection is not to exceed 10 mm. Use E = 10 GPa.



8. A girder rests on two supports 5m apart, and carries a load of 60 kN, 2 m from one support. Find the ratio of maximum deflection to deflection under the load.

9. A simply supported beam is 6 m long and has flexural rigidity of 3 MNm^2. It a carries a point load of 400 N at the middle and a UDL of 200 N/m along its entire length. Calculate slope at the ends and deflection at the middle. Prove that the relation

10. An overhanging beam ABC is loaded as shown in fig. Find the slopes over each support and at the right end. Find also the maximum deflection between the supports and the deflection at the right end. Take E= 2 $x10^5$ N/mm^2 and I= 5 x 10^8 mm^4



11. A simply supported beam, having rectangular cross-section, carries a concentrate load at the centre of span. If the maximum flexural stress is 9 N/mm^2, find the depth of section to the span ratio in order the central deflection may not exceed 1/480 of span.

12. A close helical spring 10cm mean diameter is made of 20 turns of 1 cm dia steel rod. The spring carries an axial load of 100N.find the shearing stress developed in the spring and the deflection of the load .assume modulus of rigidity 84Gpa.

13. A close-coiled helical spring is having a stiffness of 1kN/m of compression under a

maximum load of 4.5 N and a maximum shearing stress of 45 Mpa. The solid length of the spring (i.e. when the coils are touching) is to be 4.5 cm. Find the diameter of the wire and the mean diameter of the coils required. Consider G as 42 Gpa

14. a) A close-coiled helical spring made of 12 mm round steel has 12 coils and the mean diameter of the coils is 16 cm. The spring is subjected to an axial load of 150 N. Determine the elongation, intensity of tensional stress and strain energy per cubic metre under the loaded condition. G=84 Gpa.

    b) If the axial load is removed and an axial torque of 10 N-m is applied, determine the axial twist, intensity of bending stress, and work stored per cubic meter in the spring. E=210 GPa.

UNIT-II

COLUMNS AND STRUTS

Objective:

To get familiarize with different types of columns

To Analyze the crippling loads for columns for different support end conditions

To Analyze the struts for UDL and point loads.

Syllabus:

Introduction – Types of columns - Short, medium and long columns-Axially loaded compression members-crushing load- Euler's theorem for long columns-assumptions-derivation of Euler's critical load formula-various end conditions-equivalent length of column-slenderness ratio-Euler's critical stress-limitations of Euler's theory.

Learning outcome:

Student will be able to

Gain knowledge on different types of columns

Analyze and to determine the Crippling loads by using Euler's formula and Rankine's formula

Determine the Bending moments and stresses due to lateral loading on struts.

COLUMN:

Column is a vertical structural member, which is subjected to axial compressive load. It transmits the load from roof slab and beam, including its self weight to the foundation.

STRUT:

A structural member which carries an axial compressive load in roof truss is called as strut. It may be horizontal, inclined or even vertical

TYPES OF COLUMNS:

Short columns:

Columns which failed due to crushing and its slenderness ratio is less than 12 called short columns. Generally short columns are failed due to crushing loads.

Long columns:

Columns which failed due to buckling and having slenderness ratio greater than 12 are called Long columns.

Euler's Theory

Assumptions in Euler's Theory

1. Initially the column is perfectly straight and the load is applied axially
2. The cross section of column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and thus obeys Hooke's law.
4. The length of the column is very large as compared to the cross-section dimensions
5. Direct stress is neglected.
6. The failure of column occurs due to buckling alone.

End conditions of column

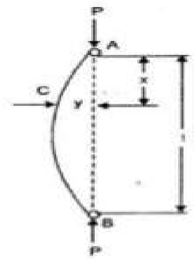
1. Both ends hinged
2. Both ends fixed
3. One end is fixed and other hinged
4. One end is fixed and other free

Note: for fixed end, the slope and deflection is zero and for hinged end, the deflection is zero

Derivation of Crippling loads for different end conditions:

- **Column hinged at both ends**

Consider a column carrying an axial compressive load P and having both ends hinged as shown in fig.



Taking origin at A, the bending moment at a distance x is

we know that

Let

General solution is

, where A and B are constants

End conditions are a) b)

At => A=0

At =>

Now B = 0 or

if B = 0 the y = 0 and the column will remain straight, which is not ture

Therefore

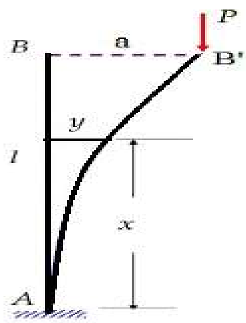
; n= 0, 1, 2, 3……

Taking fundamental value i.e., n=1

This load is known as the critical load and is denoted by P_{cr} and is also called as Euler's load

- **Columns with one end is fixed and other free**

Consider a column AB of length l fixed at A and free at B carrying a load P at B. as a result of loading the column deflect into a curved form such that the free end B deflects through 'a' and occupies a new position B_1.



Now consider any section at a distance x from A

Let y= deflection at the column at section

Moment $M_x = P(a-y)$

Let

General solution is

, where A and B are constants

End conditions are a) b) c)

At => A= -a

At => Bk = 0

Either B = 0 or k = 0

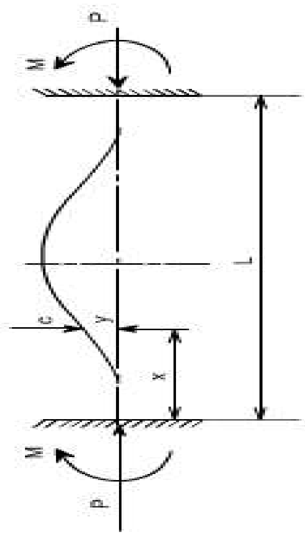
Since the load is not equal to zero. Therefore B = 0

At =>

- **Columns with both ends fixed**

Consider a column AB of length l fixed at both of its ends and carrying a critical load at B.

Now consider any section at a distance x from A



Let y= deflection at the column at section

Since both ends of the column are fixed and it is carrying a load, therefore there will be same fixed end moments at A & B.

Let M_o=FEM at A& B

Let

General solution is

, where A and B are constants

End conditions are a) b) c)

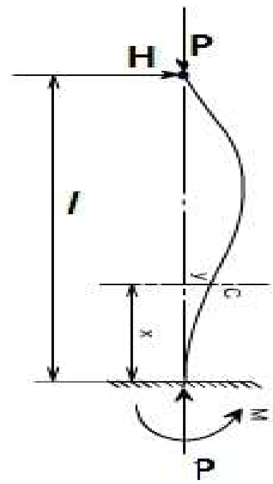
At => A = -

At => Bk = 0

B=0

At =>

- **Column with one end is fixed and other end is hinged**



Let

General solution is

, where A and B are constants

End conditions are a) b) c)

At => A =

At => Bk - = 0

B=

At =>

S.NO	END CONDITIONS	RELATION BETWEEN EFFECTIVE AND ACTUAL LENGTHS	CRIPPLING LOAD **$P_E= (\pi^2 EI) / L_e^2$**

1	Both sides hinged	L_e=L	**P_E= (π^2E I) / L^2**
2	One fixed and other free	L_e=2L	**P_E= (π^2EI) / 4L^2**
3	both fixed	L_e=L/2	**P_E= 4(π^2 EI) / L^2**
4	One fixed and other hinged	L_e=L/√2	**P_E= 2(π^2EI) / L_e^2**

- **Slenderness ratio:**

Euler's formula for the crippling load

P_E =

We know that the buckling of a column under the crippling load will take place about the axis of least moment of resistance. Subjected to I= A

A is the area, K is the least radius of gyration

P_E =

Where is known as slenderness ratio.

Slenderness ratio is defined as ratio of equivalent length of column to the least radius of gyration of the section.

- **Euler's critical stress:**

Euler's formulae for the crippling load

P_E =

Euler's critical stress

- **Limitations of Euler's formula:**

Euler's critical stress

For a column both ends hinged, $L_e = l$

Crippling stress become as Euler's critical stress , where is slenderness ratio.

If the slenderness ratio is small the crippling stress will be high. But for column material the crippling stress can't be greater than the crushing stress. Hence, when the slenderness ratio is less than a certain limit, Euler's formula gives a value of crippling stress greater than the crushing stress. In the limiting case we can find the value of *l*/k for which crippling stress is equal to crushing stress.

For example: A mild steel column with both ends hinged

Crushing stress =330N/mm^2

E=2.1x10^{5}N/mm^2

Equating the crippling stress to the crushing stress =330

Hence, if the slenderness ratio is less than 80 for mild steel column with both ends hinged, then Euler's formula will not be valid.

- **Rankine's formula:**

The empirical formula given by Rankine is given by

`

Where P= crippling load by Rankine's formula

=crushing load =

= crushing load =

Where I=A

Crippling load

- **Long columns subjected to eccentric loading:**

When a column is subjected to an eccentric load the maximum intensity of compressive stress is given by

= =

=

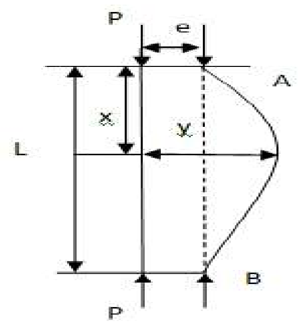
P =

When the effect of buckling is also included, then

P=

- **Column subjected to eccentric loading with both ends are hinged:**

Consider a column hinged at both ends and subjected to an eccentric load P as shown in fig.



At a distance x from A,

General solution is y=A coskx +B sinkx

x=0, y=e then A=e

x=l, y=e then e= e coskl + B sinkl.

Y=

At x=, deflection is maximum

Now

Resultant stress becomes

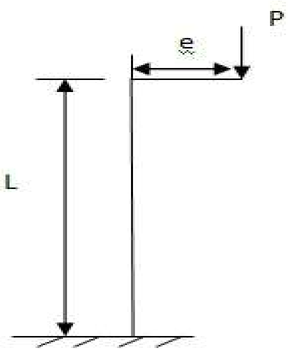
=

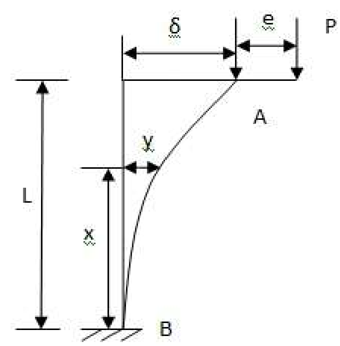
= distance of the outermost fiber in compression from the N.A

For other end conditions,

- **Column subjected to eccentric loading one end fixed other end free:**

When a column is subjected to an eccentric load P at eccentricity e. let us assume that top of the column is free and the bottom of the column is fixed.





General solution is y=A coskx + B sinkx+

At B, x=0, y=0 then A=-

X=0,,

Bk= 0

B=0

At A, x=l, y=

The max bending moment for the column at B and is equal to P

M= P.e.seck l=Pesecl

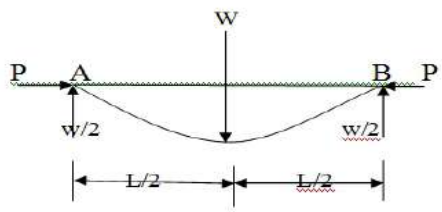
For maximum compressive stress

=

If both ends are hinged, then

- **Strut pinned at both ends and subjected to an axial thrust P and a transverse point load W at the center:**

  Consider any section at a distance x from A



General solution is y=A coskx +B sinkx -

x=0, y=0 then A=0

X=, then

Y= -

Maximum deflection:

At x= , y=

Maximum bending moment:

Substitute maximum deflection in above equation, then

Tan=

When is small

Tan=

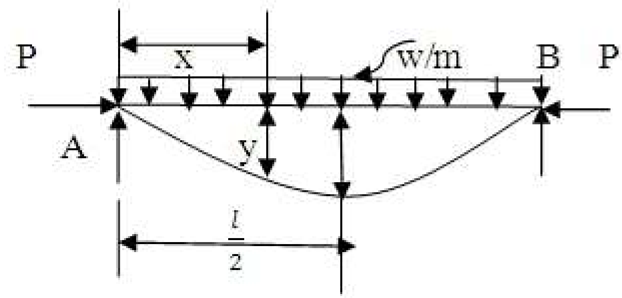
Maximum stress:

Stress due to bending, =

=

- **Strut subjected to an compressive axial thrust P and a transverse u.d.l w per unit length of both ends are pinned:**

Consider any section at a distance x from A



Differentiate the above equation is w.r.t x,

Again differentiate the above equation is w.r.t x

Solution for the above equation is

M =A coskx +B sinkx+

x=0, y=0 then A==

X=,

B=

M= coskx sinkx+

Maximum bending moment:

At X=

= coskx sinkx+

Maximum deflection:

at X=, y=, M=

Maximum stress:

UNIT – III

INDETERMINACY - PROPPED CANTILEVERS

Objective:

To learn the concept of Static and kinematic indeterminacy and analyse the propped cantilevers.

Syllabus:

Degree of static and kinematic indeterminacy- analysis of propped cantilevers for concentrated loads and UDL-shear force and bending moment diagrams

Learning Outcomes:

After completion of this unit the student will be able to

1. Distinguish between static and kinematic indeterminacy.
2. Evaluate prop reaction, shear force and bending moment for propped cantilever beam
3. Draw the shear force and bending moment diagrams for different conditions for propped cantilever and fixed beams

Learning Material

Introduction:

Structure is an assemblage of a number of components like slabs, beams, columns, walls, foundations and so on, which remains in equilibrium.

When any elastic body is subjected to a system of loads and deformation takes place and resistance is setup against the deformation, then the elastic body is known as Structures. If no resistance is setup in the body against deformation, it is known as an unstable structure or mechanism.

Classification of structures:

a) Based on type of joints:

- **Pin jointed frames:** Members are connected by means of pin joints. These frames support the loads by developing only axial forces.
- **Rigid Jointed frames:** These frames resist external forces by developing bending moments, shear forces, axial forces and twisting moments in the members of the frame.

b) Based on Dimensions:

- **Plane frames:** All members of the plane frame as well as external loads are assumed to be in one plane.
- **Space frames:** All members do not lie in one plane. Very often, it is also a combination of series of frames.

c) Based on static equilibrium conditions:

- **Determinate Structures:**

Determinate structures are analyzed just by the use of basic equilibrium equations. By this analysis, the unknown reactions are found for the further determination of stresses.

Examples of determinate structures are: cantilever beams, three hinged arches etc.

- **Indeterminate Structures:**

Redundant or indeterminate structures are not capable of being analyzed by means of use of basic equilibrium equations. Along with the basic equilibrium equations, some extra conditions are required to be used like compatibility conditions of deformations etc to get the unknown reactions for drawing bending moment and shear force diagrams.

Examples of indeterminate structures are: Propped cantilever, fixed beams, continuous beams, fixed arches, two hinged arches, portals, multi-storeyed frames, etc.

Special methods like strain energy method, slope deflection method, moment distribution method, column analogy method, virtual work method, matrix methods, etc are used for the analysis of redundant structures.

Static Indeterminacy

The number of equations required over and above the equations of static equilibrium to find the unknown reactions is known as degree of static indeterminacy or degree of redundancy of the structure.

$D_S = D_{se} + D_{si}$

D_{se} = External indeterminacy

= r-6 (for space fame)

= r-3 (for plane fame)

Here r indicates number of reactions

D_{si} = Internal Indeterminacy

= m-(2j-3), for pin jointed plane frame

= m-(3j-6), for pin jointed space frame

= 3C , for rigid jointed plane frame

= 6C , for rigid jointed space frame

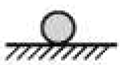
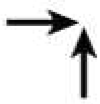
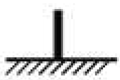
Here j indicates number of joints.

C indicates number of closed loops.

Kinematic Indeterminacy

It is defined as the number of independent components of joint displacements with respect to a specified set of axes. It is also called as degrees of freedom.

For beams the reactions and degree of freedom at an ends are as follows,

Support Type	**Image**	**Reactions**	*r*	*D.O.F*
Roller			*r*=1	*2*
Pin			*r*=2	*1*
Fixed			*r*=3	*0*

For pin jointed frames

D_k = 2j – r (for plane frames)

D_k = 3j – r (for space frames)

D_k – Degree of kinematic indeterminacy

j- Number of joints

r- Number of reactions

For rigid jointed frames

D_k = 3j – r (for plane frames considering axial strains)

Dk = 3j – (m+r), (for plane frames neglecting axial strains)

D_k = 6j – r (for space frames considering axial strains)

Dk = 6j –(m+ r) (for space frames neglecting axial strains)

D_k – Degree of kinematic indeterminacy

j- Number of joints

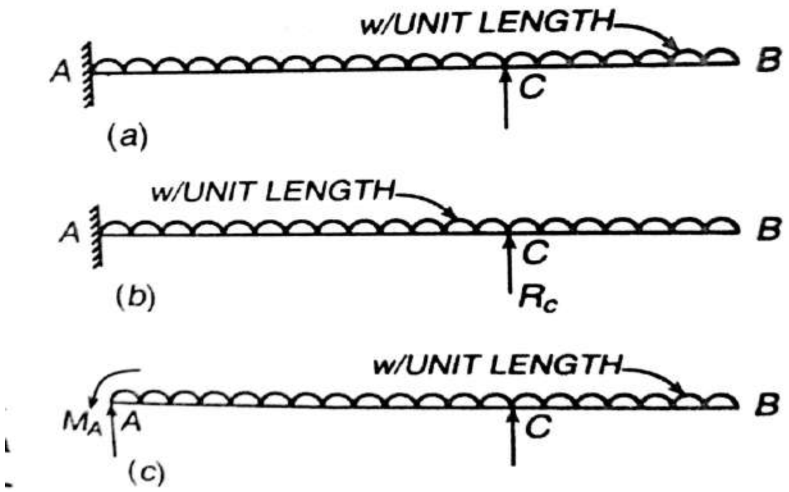
r- number of reactions

m- number of members.

Propped Cantilever

A cantilever supported at any point in the beam is called as a Propped Cantilever. When a cantilever is supported at any point in the span, the structure becomes indeterminate. Under vertical load, there will be two unknown reactions at the fixed end and one at supported end. Two equations of statics i.e. $\sum V = 0$ and $\sum M = 0$ are available. This type of structure cannot be analysed by the equations of the statics. One more equation besides two equations of statics is required to solve three unknowns. Therefore, this structure is said to be indeterminate to first degree. The third equation can be obtained by considering the deflections or slopes.

Statically indeterminate structures can be analysed by using method of consistent deformation and moment area method.



Method of consistent deformation

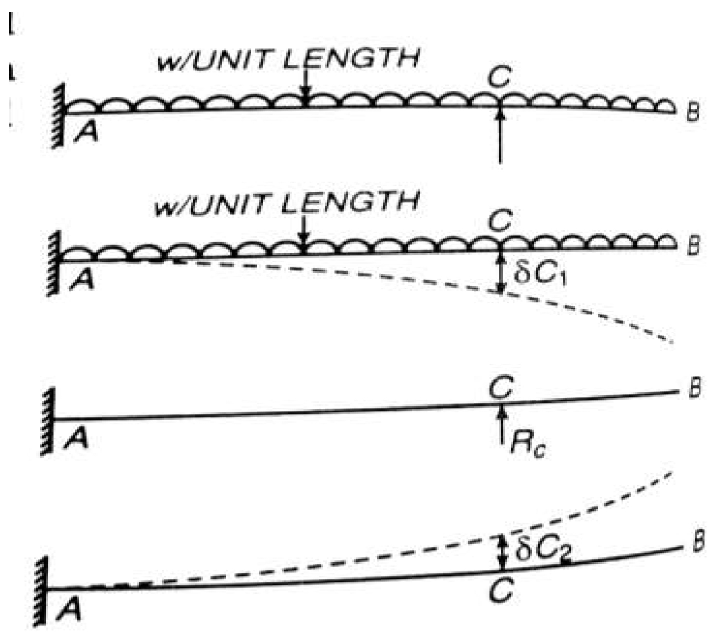
Step-1

1. In the first step the support at C is removed and the deflection at C is calculated. Let it be Δc_1.
2. The loading is removed and force R_c equal to unknown reaction at C, is applied at C and the deflection at C is worked out. Let the deflection be Δc_2. Then

   $\Delta c_1 + \Delta c_2 = 0$ in case the support C remain at the same level when the beam is loaded.

   $\Delta c_1 + \Delta c_2 = \Delta$ in case the support C sinks by Δ.

3. By using the above two equations the unknown reaction R_c can be obtained and the structure can be analyzed.

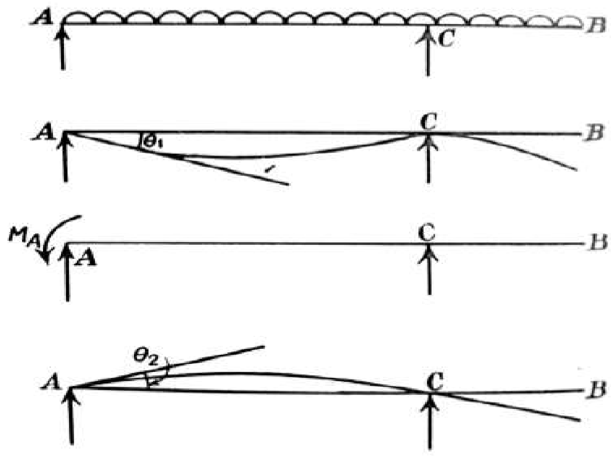


Step-2

1. The structure is made determinate by removing fixity at A and thus the structure will be a simply supported beam with overhang.
2. Under external loading the slope at A is worked out. Let the slope be θ_1.
3. The load is removed and a moment M_A equal to fixed end moment is applied at A. Let this slope be θ_2. Then

   $\theta_1 + \theta_2 = 0$, in case there is no rotation of supports.

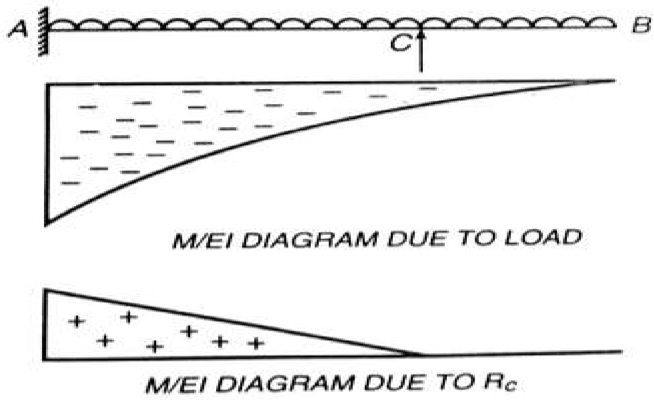
   $\theta_1 + \theta_2 = \theta$, in case the support rotates by θ.
4. By using the above equation the value of unknown moment M_A can be calculated.



Moment Area Method

Step-1 (Taking R_c as indeterminate reaction)

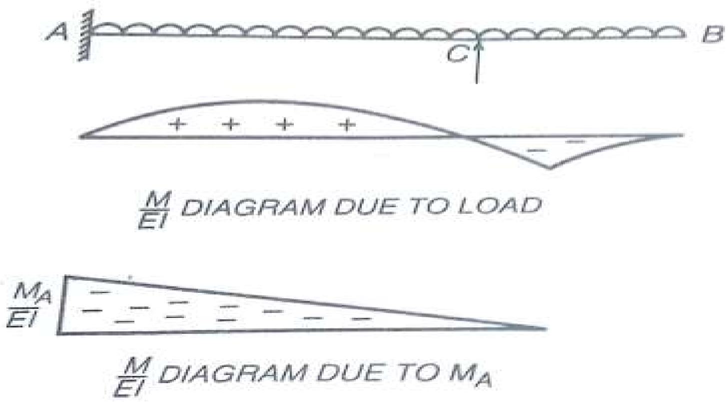
1. The bending moment diagrams due to external loading and R_c are drawn considering ABC as a cantilever.



2. As the fixed end A is fixed, the tangent to the deflection curve at A will pass through C in case A and C are at the same level. Thus the moment of M/EI diagram between A and C about C will be zero. If there is change of level equal to Δ between A and C the moment of M/EI diagram between A and C will be -Δ.

Step-2 (Taking M_A as indeterminate moment)

1. The bending moment diagrams are drawn for the load and fixed end moment M_A considering ABC as simply supported beam with overhang.



2. In case A and C are the same level, the moment of M/EI diagram between A and C about C will be zero.
3. If there is change of level Δ between the supports A and C after loading, the moment of M/EI diagrams between A and C about C will be -Δ.

Problems

1. Draw B.M diagram for the propped cantilever subjected to point load, as shown in the figure. The support A & B remain at the same level after loading.

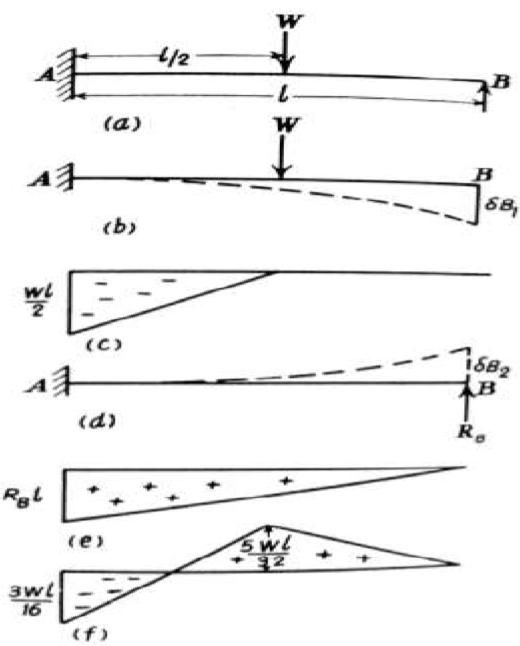
Sol: The support at B is removed and the B.M diagram for the cantilever is drawn as shown in fig C

The deflection Δ_{B1} at B will be equal to the moment of M/EI diagram about B.

Unknown reaction R_B is applied at B and Bending Moment diagram is drawn

Upward direction at B

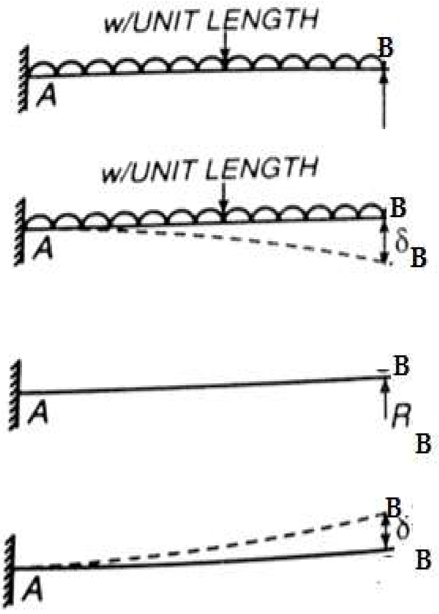
By solving



Maximum +Ve B.M will be at the centre and is equal to 5WL/32.

2. Determine the reaction components for the propped cantilever subjected to UDL as shown in figure.

   Sol: To analyse this propped cantilever method of consistent deformation is used and the deflection criteria is considered.



Remove the support at B as shown in the figure.

Step 1: Let be the deflection at point B due to external loading.

According to Moment area theorem 2

= Moment of area of M/EI diagram between A& B about B.

= $Wl^4/8EI$

Step 2: Remove the external loading and introduce unknown reaction .

R_B = Propped reaction at point B as shown in fig C.

Let be the deflection at point B due to R_B.

Then according to moment area theorem

= $R_Bl^3/3EI$

Since the supports are at the same level even after loading =

$R_B = 3WL/8$.

UNIT-IV
FIXED BEAMS

Objective:

To get familiarize with different types of fixed beams

To Analyze the loads for different beams

Syllabus:

Analysis of fixed beams for concentrated loads and UDL- SFD and BMD with and without sinking of supports.

Learning outcome:

Student will be able to

Analyze the fixed beams with and without sinking of supports.

<u>Fixed Beams</u>

Introduction

A fixed or a build in beam has both of its ends rigidly fixed so that the slope at the ends remains zero. Such a beam is also called as the encaste beam. The fixed ends give rise to fixing moments there in addition to the reactions. If perfect end fixing can be achieved, build in beams carry smaller maximum bending moments and have smaller deflections that the corresponding simply supported beams with the same loads applied. Therefore they are stronger and stiffer. However the need for high accuracy in aligning the supports and fixing the ends during erection increases the cost. Small subsidence of either support or temperature changes can set up large stresses. The end fixings are also normally sensitive to vibrations and fluctuations in bending moments.

There are four unknown reaction components. Two at end A i.e., R_A & M_A and two at end B i.e R_B & M_B. But the available equilibrium equations are two only i.e., $\sum V = 0$ and $\sum M = 0$.

Fixed beam is a statically indeterminate structure and its degree of indeterminacy is 2.

So we need two more equations to analyse the fixed beam.

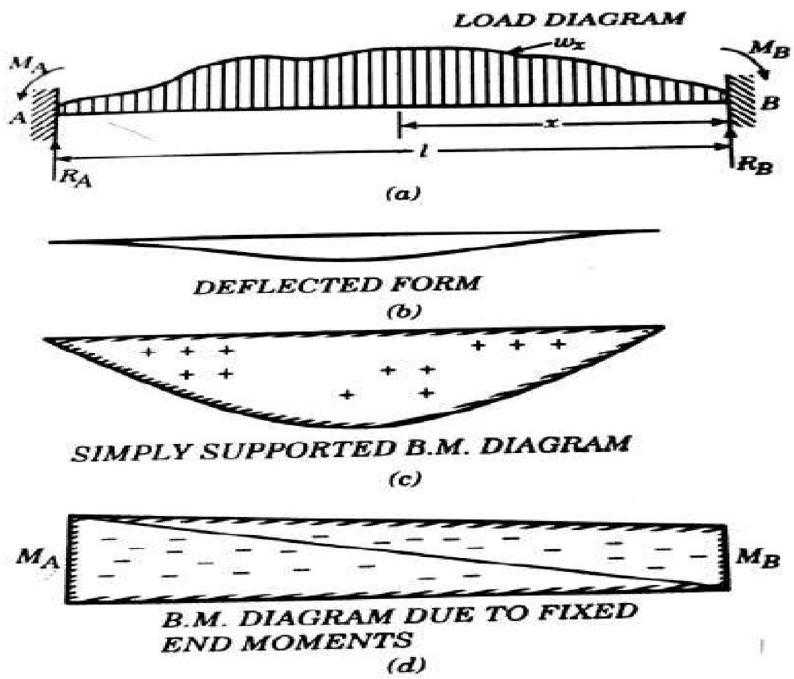


Fig (a) shows fixed beam AB of uniform section and span l loaded as shown in the figure. As the ends of the beam are fixed, the slope at support will be shown in fig (b). Let M_A and M_B be the fixed end moments at supports A & B respectively. The angle between the tangents drawn on the deflected curve is equal to zero. The area of M/EI diagram between A & B is Zero.

The fixed beam can be taken as simply supported beam with end moments M_A &M_B such that the slopes at the supports are zero. Due to simply supported condition the loading will cause +ve B.M. the B.M will vary from M_A at A to M_B B. So the area of M/EI diagram due to fixed end moments is equal to area of M/EI diagram due to simply supported beam.

Let A_s be the area of B.M considering beam as simply supported and A_i be the area of the B.M due to fixed end moments.

$A_i = A_s$

…………………..(1)

The intercept made by the tangents drawn at A & B will be zero. Therefore moment of area of M/EI diagram between A & B about B will be zero. Similarly, moment of area of M/EI diagram between A & B about B will be zero.

$$Ax/EI = 0$$

Here x is the distance of the C.G of B.M diagram area from the support.

X_s = distance of C.G of A_s from end A

X_i = distance of C.G of A_i from end A.

By substituting the values we finally get

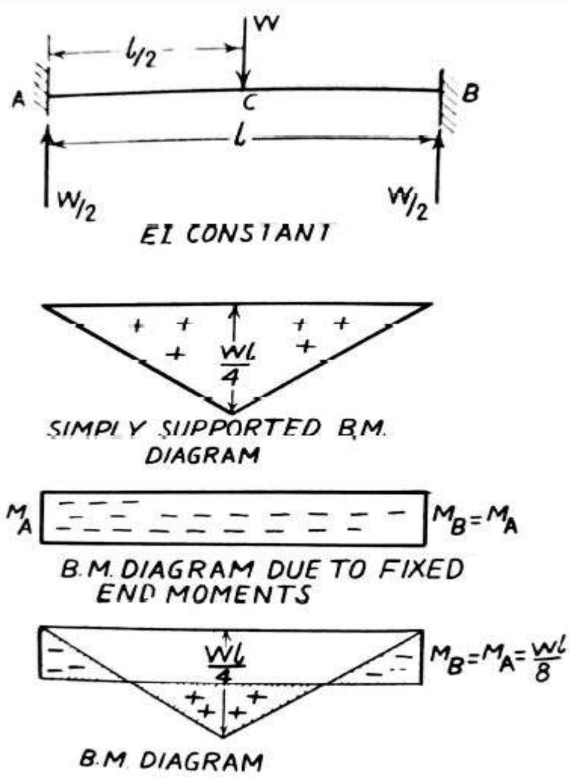
………………………(2)

From equations 1& 2 the values of and M_B can be found out.

Calculation of fixed end moments for a fixed beam of uniform section.

Case 1: Concentrated load at the centre of span.

A fixed beam can be treated as a simply supported beam with end moments M_A & M_B. So that the slope at the supports is zero.



Simply supported bending moment diagram is a triangle and bending moment diagram due to fixed end moments is a rectangle.

Since the beam is symmetrical $M_A = M_B$

We have $A_s + A_i = 0$

(

$M_A = M_B = WL/8$

To find the point of contra flexure equate M_x to Zero.

Point of contra flexure occurs at L/4 from either end.

Maximum +ve B.M occurs at the centre & is equal to WL/8

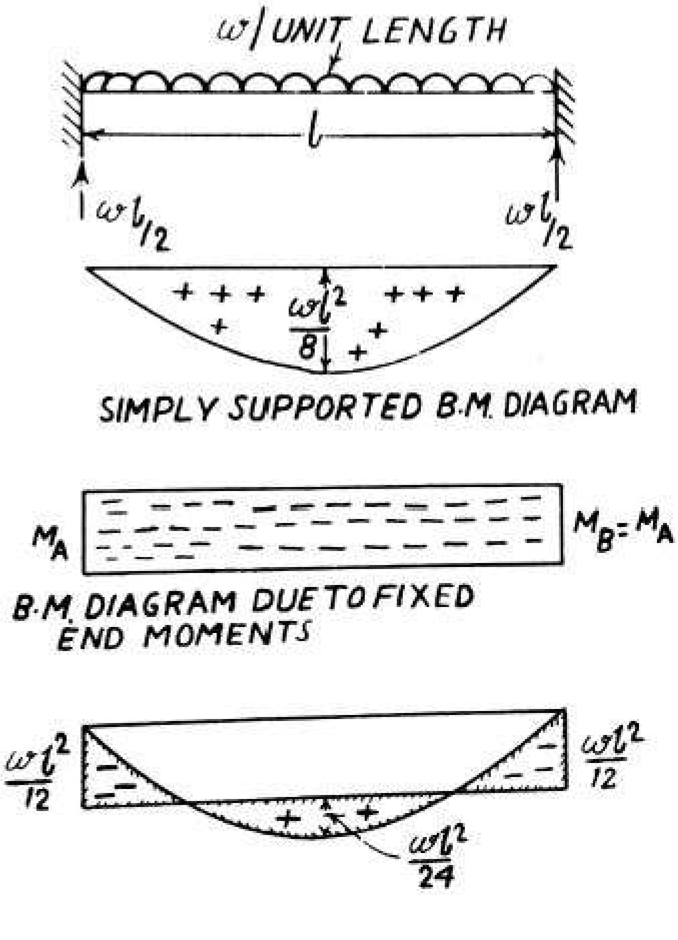
Maximum –Ve B.M occurs at the supports & is equal to -WL/8.

Case :2

UDL throughout the span.

A fixed beam can be treated as a simply supported beam with end moments M_A & M_B. So that the slope at the supports is zero.

Simply supported bending moment diagram is a triangle and bending moment diagram due to fixed end moments is a rectangle.



Since the beam is symmetrical $M_A = M_B$

We have $A_s + A_i = 0$

$M_B = Wl^2/12 = M_A$.

Point of contraflexure occurs at a distance of 0.212L from either ends.

Maximum +Ve B.M occurs at the centre & is equal to $Wl^2/24$

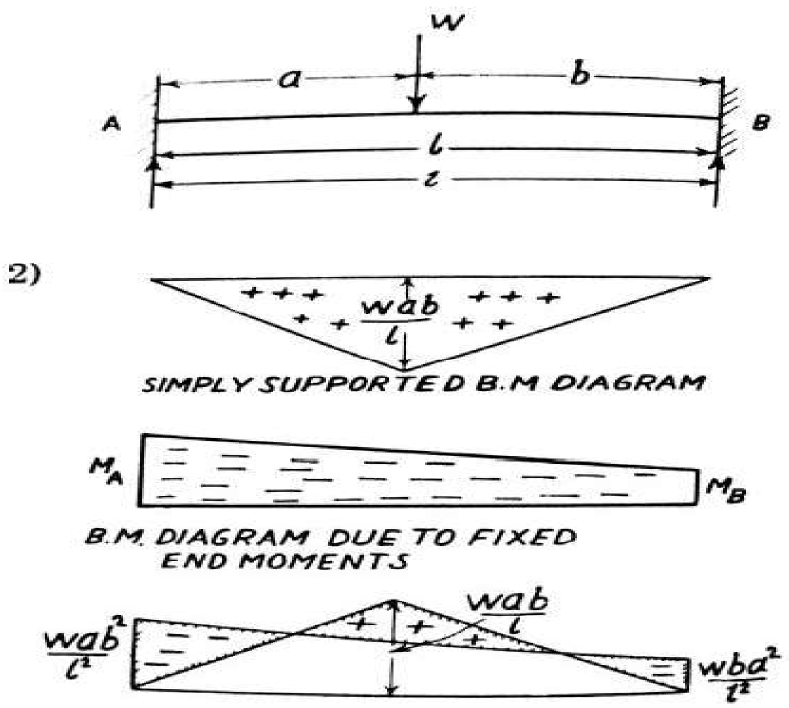
Maximum -Ve B.M occurs at the supports & is equal to $Wl^2/12$.

Case 3:

Unsymmetrical Concentrated Load.

A fixed beam can be treated as a simply supported beam with end moments M_A & M_B. So that the slope at the supports is zero.

Simply supported bending moment diagram is a triangle and bending moment diagram due to fixed end moments is a rectangle.



$R_A = Wb/l$ & $R_B = Wa/l$

Maximum B.M for simply supported beam = Wab/l

We have a relation

…………………(1)

…………………(2)

From relation (1)

…………………(3)

From relation (2)

…………………(4)

Solving equations 3 & 4

$$M_B = Wa^2b/l^2$$

$$M_A = Wab^2/l^2$$

Point of contra flexure occurs at a distance of **ab/l** from either ends.

UNIT – V

CONTINOUS BEAMS – THEOREM OF THREE MOMENTS

A continuous beam is a statically indeterminate multispan beam on hinged support. The end spans may be cantilever, may be freely supported or fixed supported. At least one of the supports of a continuous beam must be able to develop a reaction along the beam axis

Objectives:

Derive the Clapeyron's theorem of three moments Analyze continuous beam with different moment of in- ertia with unyielding supports Analyze the continuous beam with different moment of inertia in different spans along with support settlements using three moment equation.

11.1 INTRODUCTION

A beam is generally supported on a hinge at one end and a roller bearing at the other end. The reactions are determined by using static equilibrium equations. Such as beam is a statically determinate structure. If the ends of the beam are restrained/clamped/encastre/fixed then the moments are included at the ends by these restraints and these moments make the structural element to be a statically indeterminate structure or a redundant structure. These restraints make the slopes at the ends zero and hence in a fixed beam, the deflection and slopes are zero at the supports.

A continuous beam is one having more than one span and it is carried by several supports (minimum of three supports). Continuous beams are widely used in bridge construction. Consider a three bay of a building which carries the loads W_1, W_2 and W_3 in two ways.

FIG. 11a Simply supported beam

FIG. 11b Bending moment diagrams

A *B* *C* *D*

FIG. 11c Continuous beam

FIG. 11d Bending moment diagram

If the load is carried by the first case then the reactions of individual beams can be obtained by equilibrium equations alone. The beam deflects in the respective span and does not depend on the influence of adjacent spans.

In the second case, the equilibrium equations alone would not be sufficient to determine the end moments. The slope at an interior support *B* must be same on either side of the support. The magnitude of the slope can be influenced by not only the load on the spans either side of it but the entire loads on the span of the continuous beam. The redundants could be the reactions or the bending moments over the support. Clapeyron (1857) obtained the compatibility equation in term of the end slopes of the adjacent spans. This equation is called theorem of three moments which contain three of the unknowns. It gives the relationship between the loading and the moments over three adjacent supports at the same level.

11.2 DERIVATION OF CLAPEYRON'S THEOREM (THEOREM OF THREE MOMENTS)

Figure 11e shows two adjacent spans *AB* and *BC* of a continuous beam with two spans. The settlement of the supports are Δ_A, Δ_B and Δ_C and the deflected shape of the beam is shown in $A'B'C'$ (Fig. 11f).

A *B* *C*

FIG. 11e

The primary structure is consisting of simply supported beams with imaginery hinges over each support (Fig 11g). Fig 11h shows the simply beam bending moment diagrams and Fig 11i shows the support moment diagram for the supports.

A compatibility equation is derived based on the fact that the end slopes of adjacent spans are equal in magnitude but opposite in sign. Using Fig 11f and the property similar triangles

$$\frac{\Delta_B - \Delta_A + \delta^B_A}{GD} = \frac{\Delta_C - \Delta_B + \delta^B_C}{CF}$$

$$DB_j = B_jF$$

l_1 l_2

δ_B δ_B $\Delta_A - \Delta_B$ $\Delta_C - \Delta_B$

i.e. $\underline{\ A} + \underline{\ C} = \qquad + \qquad$ (i)

l_1 l_2 l_1 l_2

The displacements are obtained as follows.

(ii)

$$™_B = \frac{1}{E_1 I_1} \left(A_1 \bar{x}_1 + \frac{1}{2} M_A l_1 \cdot \frac{l_1}{3} + \frac{1}{2} M_B l_2 \cdot 2l_2/3 \right)$$

$$\delta_B = \frac{1}{+} \cdot A_2 \bar{x}_2 \cdot \frac{1}{} M_C l_2 \, l_2 + \frac{1}{} M_B l_2 \cdot 2l_2/3 \; \Sigma$$

^{C} E_2I_2 2 3 2

Combining the equations (i) and (ii)

$M\,l \quad .\,l \quad l\ \Sigma \quad l \quad .\,A\,x^{-} \quad A\,x^{-}\,\Sigma$

$^{A\ 1} + 2M_B \quad _1 + \quad _2 \quad + M_C \quad ^2 + 6 \quad _{1\ 1} + \quad _{2\ 2}$

E_1I_1 E_1I_1 E_2I_2 E_2I_2 $E_1I_1l_1$ $E_2I_2l_1$

$$= 6\left[\frac{\Delta_A - \Delta_B}{l_1} + \frac{\Delta_C - \Delta_B}{l_2}\right] \qquad \text{(iii)}$$

The above equation is called as Clapeyron's equation of three moments.

In a simplified form of an uniform beam section ($EI =$ constant); when there are no settlement of supports

$$. A \bar{x} \quad A \bar{x} \Sigma$$

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = -6 \left(\frac{\;_1{}^1}{l_1} + \frac{\;_2{}_2}{l_2} \right) \qquad \text{(iv)}$$

It is to be mentioned here that $\bar{x}_1$ and $\bar{x}_2$ are measured outwards in each span from the loads to the ends.

11.2.1 Procedure for Analysing the Continuous Beams using Theorem of Three Moments

(1) Draw simple beam moment diagram for each span of the beam. Compute the area of the above diagrams viz, A_1, A_2 ... A_n and locate the centroid of such diagrams $\bar{x}_1$, $\bar{x}_2$ ... $\bar{x}_n$. It must be re- membered that the distances $\bar{x}_1$, $\bar{x}_2$ ... $\bar{x}_n$ are the centroidal distances measured towards the ends of each span as shown in Fig. 11j.

FIG. 11j Simple beam moment diagrams

(2) Identify the support moments which are to be determined viz, M_A, M_B and M_C

(3) Apply three moment equation for each pair of spans which results in an equation or equations which are to be solved simultaneously. If the beam is of uniform section ($EI =$ constant) and no support settlements apply equation (iv) and in case the beam is non-uniform and the support settles/raises apply equation (iii).

(4) The solution of the equations gives the values of the support moments and the bending moment diagram can be drawn.

(5) The reactions at the supports and the shear force diagram can be obtained by using equilibrium equations.

11.3 APPLICATION OF THREE MOMENT EQUATION IN CASE OF BEAMS WHEN ONE OR BOTH OF THE ENDS ARE FIXED

11.3.1 Propped Cantilever Beam

Consider the propped cantilever beam of span AB, which is fixed at A and supported on a prop at B. It is subjected to uniformly distributed load over the entire span. The fixed end moment at the support A can be determined by using theorem of three moments.

A' zero span A B

FIG. 11k Propped cantilever beam

As the A is fixed support, extend the beam form A to A^j of span ‘zero length’ and A^j is simply supported.

(1) The simple beam moment diagram is a parabola with a central ordinate of $(wl^2/8)$. The centroid of this bending moment diagram (symmetrical parabola) is at a distance ‘$l/2$’ from the supports A and B.

A B

FIG. 11l Simple beam moment diagram

It's area is $A = \left(\frac{2}{3}\right)(l)\left(\frac{wl^2}{8}\right) = \frac{wl^3}{12}$.

(2) The support moment diagram is drawn as

M_A

l

FIG. 11m Pure moment diagram

(3) Apply three moment theorem for the span AB.

$$M' (0) + 2M_A(0 + l) + 0 = -6 \; . \; wl^3 \; \Sigma . \; l \; \Sigma$$

A 12 2

$\therefore$

(4) The support reactions are computed by drawing the free body diagram as

$wl^2/8$ w/m

A B

l

V_A V_B

FIG. 11n Free body diagram

$$\sum V = 0; \qquad V_A + V_B = wl$$

$$\sum M_A = 0; \quad -\frac{wl^2}{8} + \frac{wl^2}{2} - V_B l = 0$$

and hence

(5) Using the reactions, the shear force diagram and bending moment diagrams are obtained as

FIG. 11o Shear force diagram

The point of contraflexure is determined by equating the bending moment expression to zero and hence

$$\frac{5wl\,x}{8} \quad \frac{wx^2}{2} - \frac{wl^2}{8} = 0$$

$$l^2 + 4x^2 - 5lx = 0$$

Solving the above equation we get $x = l$
and

The location of maximum positive bending moment from support *A* is obtained by equating the shear force to zero.

$$\frac{5wl}{8} - wx = 0$$

At this location, the maximum positive bending moment is obtained from

$-wl^2$ $.\ 5wl\ \Sigma.\ 5l\ \Sigma$ $w(5l/8)^2$

Max + ve BM = 8 + 8 8 - 2

$$M_C = -\frac{wl^2}{8} + \frac{25wl^2}{64} - \frac{25wl^2}{128} = \frac{9wl^2}{128} = 0.07wl^2$$

$0.07\ wl^2$

A *B*

$\left(\frac{wl^2}{8}\right)$

FIG. 11p Bending moment diagram

11.3.2 Beams with Both the Ends Fixed

Consider a beam *AB* of span *l* is fixed at both the ends. The beam is carrying a concentrated load of *W* at a distance of '$l/3$' from the fixed end *A*.

As the end *A* is a fixed support, extend this *A* to A' of span (l') of zero length and is also simply supported at A'. Likewise the end *B* is extended to B'.

The simply supported bending moment diagram is drawn with the maximum ordinate as $\frac{W \times (l/3) \times (2l/3)}{l} = 2Wl/9$.

The centroid of the unsymmetrical triangle is shown in Fig. 11.3j.

W

$l/3$ $2l/3$

$l' = O$ l $l' = O$

A' A B B^1

FIG. 11q Fixed beam

FIG. 11r Simple beam moment diagram

$\binom{l+a}{3}$ —— —$\binom{l+b}{3}$

FIG. 11s Centroid of an unsymmetrical triangle

The centroid of the simply supported BMD is obtained using the above as $\frac{4l}{9}$ from A and $\frac{5l}{9}$

from *B*.

The area of the bending moment diagram is

$$\left(\frac{1}{2}\right)(l)\left(\frac{2Wl}{9}\right) = \frac{Wl^2}{9}$$

The support moment diagram can be drawn by identifying the support moments as M_A and M_B. Thus

M_A M_B

l

FIG. 11t Pure moment diagram

Applying three moment theorem for a pair of spans of A_1AB (Ref Eq (iv))

$.Wl^2\ \Sigma.\ 5l\ \Sigma$

$$M_A^{\mathbf{j}}(0) + 2M_A(0 + l) + M_B(l) = 0 - 6 \quad 9 \quad 9 \times 1/l$$

$$2M_A + M_B = -0.37\ Wl$$

Considering the next pair of spans ABB'

$$.\frac{Wl^2}{}\Sigma.\ \frac{4l}{}\Sigma$$

$$M_A l + 2M_B(l + 0) + M' (0) = -6$$

Free body diagram to determine the reactions

0.148 Wl $l/3$ W $2l/3$ 0.074 Wl

C

V_A V_B

FIG. 11u

Using the static equilibrium;

$$\sum V = O; \qquad V_A + V_B = W$$

$$. \, l \, \Sigma$$

$$3$$

$$\sum M_A = O; \qquad -0.148\ Wl + \qquad - V_B l + 0.074\ Wl = O$$

$$W$$

0.74 W

0.26 W

FIG. 11v Shearforce diagram

0.0986 Wl

– 0.148 Wl – 0.074 Wl

FIG. 11w Bending moment diagram

11.4 NUMERICAL EXAMPLES ON CONTINUOUS BEAMS

EXAMPLE 11.1: A continuous beam *ABC* is simply supported at *A* and *C* and continuous over support *B* with $AB = 4m$ and $BC = 5m$. A uniformly distributed load of 10 kN/m is acting over the beam. The moment of inertia is *I* throughout the span. Analyse the continuous beam and draw *SFD* and *BMD*.

A *C*

FIG. 11.1a

20 31.25 kNm

A *C*

— $-x_1$ — $-x_2$

FIG. 11.1b Simple beam moment diagram

FIG. 11.1c Pure moment diagram

Properties of the simple beam BMD

$A_1 = 2 \times 4 \times 20 = 53.33$ kNm^2 $A_2 = 2 \times 5 \times 31.25 = 104.17$ kNm^2

3

$\bar{x}_1 = 2\text{m}$

$l_1 = 4\text{m}$

Applying three moment equation for the span *ABC*

3

$\bar{x}_2 =$ 2.5m

$l_2 =$ 5.0m

$A\bar{x}$ Σ

. $A\bar{x}$

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = -6 \quad l_1 {}^{1\,1} + {}^{2\,2} l_2$$

$$2M_B(4 + 5) = -6\left(\frac{53.33 \times 2}{4} + \frac{104.17 \times 2.5}{5}\right)$$

$$18M_B = -6(26.67 + 52.1)$$

$$M_B = -26.26 \text{ kNm.}$$

EXAMPLE 11.2: Analyse the continuous beam by three moment theorem. Draw *SFD* and *BMD*.

10 kN 10 kN

2 4 m 3 3 m

D 6 m *E* 6 m

A *B* *C*

FIG. 11.2a

Solution

The simple beam moment diagram is drawn as

$$M_D = Wab/l = \frac{10 \times 2 \times 4}{6} = 13.33 \text{ kNm}$$

$$M_E = Wl/4 = \frac{10 \times 6}{4} = 15 \text{ kNm}$$

A *D* *B* *E* *C*

FIG. 11.2b Simple beam moment diagram

A *B* *C*

FIG. 11.2c Pure moment diagram

Properties of the simple beam BMD

A 1 1

$-40 = 0.2\ (6)\ 13.33 =$

$\bar{x}_1 = \frac{6 + 2}{3} = 2.67$ m

$l_1 = 6$m

$A_2 = 2 \times 6 \times 15 = 45$

$\bar{x}_2 = 3$m

$l_2 = 6$m

A *B* *C*

FIG. 11.4b Simple beam moment diagram

A *C*

FIG. 11.4c Pure moment diagram

Properties of the simple beam BMD

$A_1 = 2 \times 5 \times 50 = 167.5\ \text{kNm}^2$ $A_2 = 1 \times 10 \times 60 = 300\ \text{kNm}^2$

3

$x_1 =$
2.5m

$l_1 = 5.0\text{m}$

2

$x_2 = 10 + 6 = 5.33$ m
3

$l_2 = 10$ m

$$5M_A + 2M_B(5 + 10) + 10M_C = -6\left[\frac{167.5 \times 2.5}{5.0} + \frac{300 \times 5.33}{10}\right]$$

$$30\,M_B = -6\,(83.75 + 159.9)$$

Properties of the simple beam BMD

$\sum V = 0; \quad V_A + V_{B1} = 80$ (i) $\qquad V_{B2} + V_C = 25$ (iii)

$\sum M = 0; \quad 5V_A + 49 - \frac{16(5)^2}{2} = 0 \quad \text{(ii)}$

$10V_{B2} - 25(6) - 49 = 0 \text{(iv)}$

$V_A = 30.2$ kN
$V_{B1} = 49.8$ kN

$V_{B2} = 19.9$ kN
$V_C = 5.1$ kN

The simple beam moments are

$$M_D = 20 \times 10^2/8 = 250 \text{ kNm}$$

$$M_E = \frac{50 \times 6 \times 2}{8} = 75 \text{ kNm}$$

A *D* *B* *E* *C*

FIG. 11.5b Simple beam moment diagram

Properties of simple beam BMD

$A_1 = 2 \times 10 \times 250 = 1666.7 \text{ kNm}^2$

$A_2 = 1 \times 8 \times 75 = 300 \text{ kNm}^2$

3

$x_1 = 5\text{m}$

$l_1 = 10\text{ m}$

2

$x_2 = 8 + 2 = 3.33\text{ m}$

3

$l_2 = 8.0\text{m}$

Since *A* is fixed imagine a span *A*j*A* of zero length and *A*j as simply supported. Apply three moment theorem for the spans *A*j*AB*.

$$M_A(0) + 2M_A(0 + 10) + M_B(10) = -6\left(\frac{1666.7 \times 5}{10} + 0\right)$$

$$20M_A + 10M_B = -5000$$

$$2M_A + M_B = -500 \tag{i}$$

Apply three moment theorem for the spans ABC.

$M_A(10) + 2M_B(10 + 8) +$ $8M_C$ $= -6$ $. 1666.7 \times 5$ 10 $300 \times 3.33 \Sigma$ 8

$+$

$$10M_A + 36M_B = -6(833.35 + 124.875)$$

$$10M_A + 36M_B = -5749.35 \quad \text{(ii)}$$

Solving equations (i) and (ii)

Free body diagram of spans *AB* and *BC*

16 kNm

4 kN/m

B

C

A

V

V

V_A V_{B1}

FIG. 11.7d

$B2$ C

FIG. 11.7e

Static equilibrium of AB

$$\sum V = 0; \quad V_A + V_{B1} = 24 \qquad \text{(i)}$$

$$\sum M = 0; \quad 4V_A + 16 - 10 - 48 = 0 \qquad \text{(ii)}$$

Static equilibrium of BC

$$V_{B2} + V_C = 24 \qquad \text{(iii)}$$

$$\sum M_B = 0;$$

$V_A = 10.5$ kN.

$V_{B1} = 13.5.$

$$16 + 6V_C - 4 \times \frac{6^2}{2} \quad 0$$

$V_C = 9.3$ kN.

$V_{B2} = 14.7$ kN.

10.5

+ 9.3

+ x

A B D – C

–

14.7 kN

4 m 13.5 6 m

FIG. 11.7f Shear force diagram

Shear force diagram

The zero shear location in span *BC* is

$$14.7 - 4x = 0$$

$$x = 3.67 \text{ m}.$$

$$\therefore \quad \text{Maximum +ve BM} = 14.7(3.67) - 4(3.67)^2/2 - 16 = 11 \text{ kNm}$$

24

10 − + 16 kNm − 11 +

A *B* 3.67 m *C*

FIG. 11.7g Bending moment diagram

[redacted]

$$M_A - 12M_A = -60$$

Free body diagram of span *AB* and *BC*

5.45 kNm 10.9 kNm

A *B* *B* *C*

A V_{B1}

FIG 11.8d

V_{B2} V_C

FIG 11.8e

Static equilibrium of span AB

$\sum V = 0;$

$V_A + V_{B1} = 0$ (i)

$\sum M_B = 0;$

Static equilibrium of span BC

$\sum V = 0$

$V_{B2} + V_C = 60$ (iii)

$\sum M_B = 0;$

$$5.45 + 10.9 + 2V_A = 0 \qquad \text{(ii)}$$

$$-10.9 + 2V_{B2} - \frac{30 \times 2^2}{2} = 0$$

35.5 kN/m

8.2 kN 24.5

FIG. 11.8f Shear force diagram

10.9

9.55

–

A

\+ *B* *D* C

5.45 kNm

FIG. 11.8g Bending moment diagram

Applying three moment theorem for the span *ABC*

. 5 Σ . 5 6 Σ 6

M_A I $+$

$2M_B$ $I +$ $- 30 \times 1.5I$

$1.5I$

. 240 ✖ 2.67 360 ✖ 3 Σ

$$= -6 \quad 5I \quad + 6 \times 1.5I$$

[redacted]

Shear forces and moments in members *AB* and *BC*.

Member AB

80 kN

3 m 2 m 66.67 kNm

D

33.76 kNm

V_{AB} V_{BA}

FIG. 11.9d

$$\sum V = 0; \qquad V_{AB} + V_{BA} = 80 \tag{i}$$

$$\sum M_B = 0; \qquad 5V_{AB} + 66.67 - 33.76 - 80(2) = 0 \tag{ii}$$

$$V_{AB} = 25.42$$

$$\therefore \quad V_{BA} = 54.58$$

$$M_D = -33.76 + 25.42(3) = 42.5 \text{ kNm.}$$

Member BC

66.67 kNm

30 kNm

FIG. 11.9e

FIG. 11.9i Elastic curve

EXAMPLE 11.10: A continuous beam *ABCD* is simply supported at *A* and continuous over spans *B* and *C*. The span *AB* is 6 m and *BC* are of length 6 m respectively. An overhang *CD* is of 1 metre length. *A* concentrated load of 20 kN is acting at 4 m from support *A*. An uniformly distributed load of 10 kN/m is acting on the span *BC*. *A* concentrated load of 10 kN is acting at *D*.

20 kN

4 m 2 m

E

A 6 m

B

10 kN/m 10 kN

F 1 m D

6 m C

FIG. 11.10a

The simple beam moments are

$$M_E = \frac{20 \times 4 \times 2}{6} = 26.7 \text{ kNm}$$

6_2

$$M_F = 10 \times 8 \;=\; 45.0 \text{ kNm}$$
$$M_C = -10 \times 1 = -10 \text{ kNm}$$

A *E* *B* *F* *C*

FIG. 11.10b Simply suppored BMD

A *B* *C* *D*

FIG. 11.10c Pure moment diagram

Considering spans ABC

Properties the simple beam BMD

$A_1 = 1 \times 6 \times 26.7 = 80.1$ kNm^2

$A_2 = 2 \times 6 \times 45 = 180$ kNm^2

2

$x_1 = 6 + 4 = 3.33$

m.

3

$l_1 = 6$ m.

3

$x_2 = 3$ m.

$l_2 = 6$ m.

$$. \; 6 \qquad 6\,\Sigma \qquad (-10) \qquad . \; 80.1 \times 3.33 \qquad 180 \times 3\,\Sigma$$

$$+ \; 2M_B \; I + 2I \; + -6M_C \; 2I \; = \qquad 6 \qquad 6 \times 2$$

$$-6$$

$$+$$

$$18M_B - 30 = -6\,(44.45 + 45)$$

$$M_B = 28.15 \text{ kNm.}$$

Shear force and bending moment values for the spans *AB* and *BC*

20 kN

28.15 kNm

4

2

A

E

B

V_{AB}

V_{BA}

FIG. 11.10d

Using equilibrium conditions;

$$\sum V = 0; \quad V_{AB} + V_{BA} = 20 \qquad \text{(i)}$$

$$\sum M = 0; \quad 6V_{AB} + 28.15 - 20(2) = 0 \qquad \text{(ii)}$$

$$\therefore \qquad M_E = V_{AB}(4) = 7.9 \text{ kNm}$$

28.15 kNm

10 kN/m

10 kNm

$$\sum V = 0;\ V_{BC} + V_{CB} = 10(6) = 60 \quad \text{(iii)}$$

$$\sum M_C = 0 \quad 10 - 28.15 + 6\,V_{BC} - 10 \times \frac{6^2}{2} = 0 \quad \text{(iv)}$$

18.02

FIG. 11.10f Shear force diagram

7.9

28.15

+ 26

10 kNm

+ – –

A *B* *E* *C* *D*

FIG. 11.10g Bending moment diagram

EXAMPLE 11.11: Analyse the continuous beam shown in figure by three moment theorem. Draw *SFD & BMD*.

40 kN 20 kN

30 kN

2 2

4 2

E F 2 m D

A B C

FIG. 11.11a

Solution

The simple beam moments at E and F are

$$M_E = \frac{Wab}{l} = \frac{30 \times 4 \times 2}{6} = 40 \text{ kNm}$$

$$M_F = \frac{Wl}{4} = \frac{40 \times 4}{4} = 40 \text{ kNm}$$

A *E* *B* *F* *C* *D*

FIG. 11.11b Simply supported beam BMD

A *B* *C* *D*

FIG. 11.11c Pure moment diagram

Properties of simply supported beam BMD

$A_1 = 1 \times 6 \times 40 = 120 \text{ kNm}^2$ $A_2 = 1 \times 4 \times 40 = 80 \text{ kNm}^2$

2

$x_1 = 6 + 4 =$

3.33

3

$l_1 = 6.00$

2

$x_2 = 2$ m.

$l_2 = 4.0$ m.

Applying three moment theorem for spans AB & BC

. 1 $\quad$ 3.3 $\quad$ 80(2) Σ

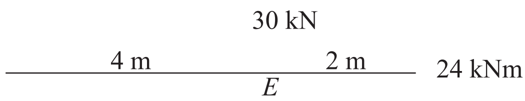
3

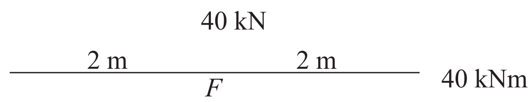
$$6M_A + 2M_B(6 + 4) + 4M_C = \quad 2 \times 6 \times 40 \times 6.00 + \ 4$$

-6

$$20M_B - 160 = -6\,(66.6 + 40)$$
$$20M_B = -479.6$$

Free Body diagrams


30 kN
4 m
2 m
24 kNm
E


40 kN
2 m
2 m
40 kNm
F

V_A V_{B1}

FIG. 11.11d

V_{B2} V_C

FIG. 11.11e

20

6

24

24 kN

FIG. 11.11f Shear force diagram

40 40 40 kNm

+ –24 –

Bending moment diagram

A *E* *B* *F* *C* *D*

FIG. 11.11g Bending moment diagram

EXAMPLE 11.12: Draw the shear force diagram and bending moment diagram for the beam shown in figure.

10 kN/m

C

FIG. 11.12a

A B C

FIG. 11.12b Simply supported beam BMD

SOLUTION

As the end *A* is fixed, imagine an imaginery span $A'A$ of zero length with no load and A' is simply supported.

Considering the span $A'AB$

j

0 + . 2 Σ

3 ✖ 3 ✖ 11.25 1.5

$$M_A(0) + 2M_A(0 + 3) + 3M_B = -6 \quad \text{(i)}$$

3

$$6M_A + 3M_B = -67.5$$

Considering the span ABC

$$3M_A + 2M_B(3 + 6) + 6M_C = -6\left[\frac{22.5 \times 1.5}{3} + \frac{180 \times 3}{6}\right]$$

$$3M_A + 18M_B = -6\,(11.25 + 90)$$

$$3M_A + 18M_B = -607.5 \qquad \text{(ii)}$$

[redacted]

The *BMD* is drawn using the above end moments as

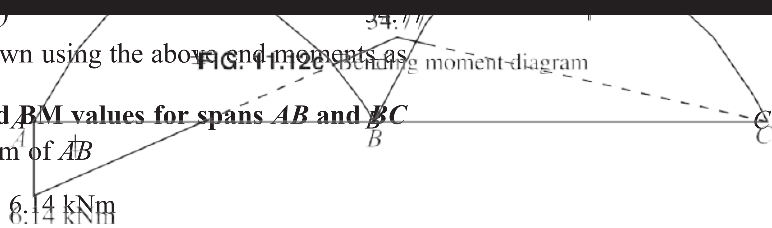


FIG. 11.12c Bending moment diagram

Shear force and BM values for spans *AB* and *BC*

Static equilibrium of *AB*

FIG. 11.12c Bending moment diagram

6.14 kNm

10 kN/m

34.77 kNm

$$\sum \; {}_{B} = 0; \quad 34.77 + 6.14 + 3V_{AB} - 10 \times 2 = 0 \qquad \text{(ii)}$$

Static equilibrium of BC

FIG. 11.12e

[redacted]

Maximum positive *BM* is span *AB*

34.77 kNm

The location of zero shear force in *AB* zone is

6.28

29.3

+

$$1.36 - 10\bar{x}_1 = 0.$$

+

$$x_1 = 0.135 \text{ m}$$

A *B* *E* *C*

+

35.8

2.42 m

6.14

FIG. 11.12f Shear force diagram

FIG. 11.12g Bending moment diagram

$$M_{X1X1} = 6.19 + 1.35\,(0.135) - 10\,(0.135)^2/2 = 6.28 \text{ kNm}.$$

Maximum positive *BM* in span *BC*

The location of zero shear force in *BC* zone is

$$24.2 - 10\,x_2 = 0$$

$$x_2 = 2.42 \text{ m}$$

$$M_{x_2 x_2} = 24.2(2.42) - 10(2.42)^2/2$$

$$= 29.3 \text{ kNm.}$$

EXAMPLE 11.13: A continuous beam *ABCD* is of uniform section. It is fixed at *A*, simply supported at *B* and *C* and *CD* is an overhang. $AB = BC = 5$ m and $CD = 2$ m. If a concentrated load of 30 kN acts at *D*, determine the moments and reactions at *A*, *B* and *C*. Sketch the shear force and bending moment diagram and mark in the salient values.

30 kN

A' *A* *B* *C* *D*

FIG. 11.13a

SOLUTION

As the end *A* is fixed imagine a imaginery span $A'A$ of zero length and A' is simply supported.

Apply three moment theorem for the spans $A'AB$

$M_{A(0)}$ $+ 2M_A(0 + 5) + 5M_B$

$= -6(0 + 0)\ 10M_A +$

$5M_B = 0$

$$2M_A + M_B = 0 \quad \text{(i)}$$

Shear force and BM values for spans *AB* and *BC*

Span AB

8.57 kNm

17.14 kNm

15.43

FIG. 11.13d Shear force diagram

8.57

17.10 kNm

FIG. 11.13e Bending moment diagram

EXAMPLE 11.14: Analyse the continuous beam by the theorem of three moments. Draw neat sketches of *SFD* and *BMD*. Clearly indicate all the salient values.

A′ *A* *C* *D* *E*

FIG. 11.14a

SOLUTION

The simple beam moments are

wl^2 wl 4^2 4

M

$_B = 8 + 4 = 20 \times 8 + 16 \times 4 = 56$ kNm

$$M_D = \frac{wl}{4} = \frac{40 \times 4}{4} = 40 \text{ kNm}$$

A *B* *C* *D* *E*

FIG. 11.14b Simple beam moment diagram

A *B* *C* *E*

FIG. 11.14c Simple beam moment diagram

Properties of simple beam *BMD*

$A_1 = 2 \times 4 \times 40 + 1 \times 4 \times 16 = 138.67$ $A_2 = 1 \times 4 \times 40 = 80\ \text{kNm}^2$

3 2

$l_1 = 4\text{m}$

$x_1 = 2\text{m}$

2

$x_2 = 2\text{m}$

$l_2 = 4\text{m}$

Applying three moment theorem for span AjAC

$$M_A l_1 + 2M_A (l_1 + l_2) + M_C l_2 = -6\left(\frac{A_1 x_1}{l_1} + \frac{A_2 x_2}{l_2}\right)$$

$$2M_A(4) + 4M_C = -6 \times 138.67 \times \frac{2}{4}$$

$$8M_A + 4M_C = -416.01 \qquad \text{(i)}$$

Applying theorem of three moments for the spans ACE

$$M_A(4) + 2M_C(4+4) + M_E(4) = -6\left[\frac{138.67 \times 2}{4} + \frac{80 \times 2}{4}\right]$$

$$4M_A + 16M_C = -6(109.335) = -656.01 \quad \text{(ii)}$$

Solving Equations (i) and (ii)

Free body diagram

16 kN 20 kN/m 32 kNm 32 kNm 40 kN

36 kNm 2 m 2 m

4 m 4 m

V V_{C2} V_E

V

A *B* *C1* *C* *D* *E*

A

FIG. 11.14d

C

FIG. 11.14e

Static equilibrium of AC		*Static equilibrium of CE*	
$\sum V = 0;$		$\sum V = 0;$	
$V_A + V_{C1} = 16 + 4(20) = 96$	(i)	$V_{C2} + V_E = 40$	(iii)
$\sum M_A = 0$		$\sum M_E = 0;$	
$-36 + 32 + 4V_A - 16(2) - 20 \times \frac{4^2}{2} = 0$	(ii)	$-32 + 4V_{C2} - 40(2) = 0$	(iv)
$V_A = 49$ kN		$\therefore \quad V_{C2} = 28$ kN	
$V_{C1} = 47$ kN		$V_E = 12$ kN	

49 56 28 40 kNm

\+ + 9 + +

36 B D E

A 32 C

– 7 – – –

47 kN 12

FIG. 11.14f Shear force Diagrams

A B C D E

FIG. 11.14g Bending moment diagram

Bending moment diagram

EXAMPLE 11.15: Sketch the BMD for the continuous beam shown in figure.

60 kN 20 kN/m 30 kN

1 m 3 m

4 m 1 m

O

A′ *A* *D* B *E* 4*I* C *F*

FIG. 11.15a

A *D* *B* *E* *C*

FIG. 11.15b Simple beam moment diagram

M_B

M_A

A *B* *C*

FIG. 11.15c Pure moment diagram

SOLUTION

Properties of the simple beam *BMD*

$A_1 = \frac{1}{2} \times 4 \times 45 = 90$ kNm^2

$x_1 = \frac{4 + 1}{3} = 1.67$ m

$l_1 = 4$m

$A_2 = \frac{2}{3} \times 4 \times 40 = 106.7$ kNm^2

$x_2 = 2$m

$l_2 = 4$m

Since A is fixed assume an imaginery span of $A'A$ of zero length with no loading. Assume A' as simply supported. Apply three moment equation for the span $A'AB$,

$$M_A (0) + 2M_A \; 0 + 4 + M_B \; 4 = -6 \; 0 + 90 \times 2.33 \times 3I$$

$$8M_A + 4M_B = -315 \quad \text{(i)}$$

Applying three moment theorem for the spans *AB* and *BC*;

. 4 Σ . 4 4 Σ . 4 Σ . 90 ✖ 1.67 106.7 ✖ 2 Σ

M_A 3I + 2M_B 3I + 4I + M_C 4I = −6 4 × 3I 4 × 4I .

+

$$1.33M_A + 2M_B(1.33 + 1.0) - 30 = -6\,(12.525 + 13.338)$$

$$1.33M_A + 4.66M_B = 30 - (25.863)6$$

$$1.33M_A + 4.66M_B = -125.18 \qquad \text{(ii)}$$

Solving (i) and (ii);

Free body diagrams of span *AB* and *BC*

30.3 kNm

60 kN

18.1 kNm

18.1 kNm

20 kN/m

30 kNm

1 m 3 m *B* *C*

A V *V B* *V* 4 m *V*

A *B1* *B2* *C*

FIG 11.15d

FIG 11.15e

Static equilibrium of AB

$\sum V = 0;$

$V_A + V_B = 60$ (i)

$\sum M_B = 0;$

Static equilibrium of BC

$\sum V = 0;$

$V_{B2} + V_C = 80$ (iii)

$\sum M_B = 0$

$-30.3 + 18.1 + 4V_A -$

$60(3) = 0 \; V_A = 48.05\ \text{kN}$ (ii)

$V_{B2} = 11.95\ \text{kN}$

4_2

$-18.10 + 30 + 4V_{B2} - 20 \times 2 = 0$

$V_{B2} = 37.02\ \text{kN}.$

$V_C = 42.98\ \text{kN}.$

48.05

37.02

30

+ + +

A D – B 1.85 E – C F

11.95

42.98 kN

FIG. 11.15f Shear force diagram

+ 15.95

30.3 + 18.1 30.0 kNm

– 45.0 –

–

A D B E C

F

FIG. 11.15g Bending moment diagram

EXAMPLE 11.16: Analyse the continuous beam by three moment theorem. *E* is constant. Draw the bending moment diagram.

80 kN 1 m 20 kN/m

2 m

A' 3 m E 4 m F 1 m D

A $3I$ B $2I$ C $2I$

FIG. 11.16a

53.33 40 kNm

A *E* *B* *F* *C* *D*

FIG. 11.16b Simple beam BMD

M_B

M_A

A *B* *C* *D*

FIG. 11.16c Pure moment diagram

SOLUTION

As the end *A* is fixed assume an imaginery span *A*′ of zero length with no load and *A*′ is simply supported;

Apply three moment theorem for spans A′AB

$. \; 3 \; \Sigma \qquad .$

$80 \times 1.33 \; \Sigma$

$3 \qquad 3I \quad = -6\;0 + $

$M_A^{j} \qquad + \; 2M_A \;\; 0 \qquad + \; 3I \quad + \quad _B \qquad 3 \times 3I$

$3I$

$$2M_A + M_B = -70.93 \quad \text{(i)}$$

Applying three moment theorem for spans ABC

. 3 Σ . 3 4 Σ . 4 Σ . 80 ✖ 106.67 ✖ 2 Σ

1.67

M_A $3I$ $+$ $2M_B$ $3I +$ $-$ 10 $=$ -6 $3 \times 3I$ $2I \times 4$

$2I$ $2I$

$+$

$$M_A + 6M_B - 20 = -6(14.84 + 26.67)$$

$$M_A + 6M_B = -249.06 \tag{ii}$$

[redacted]

Free Body diagrams of span *AB* and *BC*

16.05 kNm 80 kN 38.84 kNm 20 kN/m 10 kNm

2 m 1 m 38.84 kNm

V_A E V_{B1}

FIG. 11.16d

V_{B2} 4 m V_C

FIG. 11.16e

$$\sum V = 0;$$

$$V_A + V_{B1} = 80 \quad \text{(i)}$$

$$\sum M_B = 0$$

$$\sum V = 0;$$

$$V_{B2} + V_c = 80 \quad \text{(iii)}$$

$$\sum M_C = 0$$

$$-16.05 + 38.84 - 80(1) + 3V_A = 0 \quad \text{(ii)}$$

$$10 - 38.84 + 4V_{B2} - \frac{20 \times 4^2}{2} = 0 \quad \text{(iv)}$$

$$M_E = -17.86 + 20.88(2)$$
$$= 23.9 \text{ kNm}$$

The location of zero shear in zone *BC* is obtained from

$$47.21 - 20x = 0$$
$$x = 2.36 \text{ m}$$

$$\therefore \text{Max +ve BM} = -38.84 + 47.21 \times 2.36 - 20 \times \frac{2.36^2}{2}$$

$$= 16.88 \text{ kNm}$$

At the midspan of *BC*;

$$M_F = -38.84 + 47.21 \times 2 - 20 \times \frac{2^2}{2} = 15.58 \text{ kNm}$$

19.07
47.21
+
20
+
+
A *E* *B* *C* *D*
–
–
60.93
33.79

FIG. 11.16f Shear force diagram

53.33
15.58
+
38.84
+
16.05
10 kNm
–
–
–
A 2 m *E* 1 m *B* 2.32 m *C* 1 m *D*

FIG. 11.16g Bending moment diagram

EXAMPLE 11.17: A continuous beam *ABC* is fixed at *A* and *C*. It is continuous over a simple support *B*. Span *AB* is 5 m while *BC* span is 6 m. It is subjected to a concentrated load of 60 kN at 3 m from *A* and the span *BC* is subjected to uniformly distributed load of 10 kN/m. The ratio of flexural rigidity of span *BC* to *BA* is 1.5. Sketch the shear force and bending moment diagram. Use Clapeyron's theorem of three moments.

60 kN

10 kN/m

3 m

2 m

5 m 6 m

A′ *A* *B* *C* *C*

FIG. 11.17a

SOLUTION

The simple beam moments are

$$M_D = Wab = 60 \times 3 \times 2 = 72 \text{ kNm}$$

l 5

$$M_E = \frac{wl^2}{8} = \frac{10 \times 6^2}{8} = 45 \text{ kNm}$$

FIG. 11.16f Shear force diagram

72 45 kNm

A D B E C

FIG. 11.17b Simple beam BMD

M_B

M_A M_C

A B C

FIG. 11.17c Pure moment diagram

Since A is fixed imagine a span of zero length $A'A$ with no load and A' is simply supported.

Apply three moment theorem for the spans $A'AB$

Properties of the simple beam *BMD*

$A_1 = 0$	$A_2 = \frac{1}{2} \times 5 \times 72 = 180$
$x_1 = 0$	$x_2 = \frac{5+2}{3} = 2.33$
$l_1 = 0$	$l_2 = 5.0$

$$M_j \, l_1 + 2M_A (l_1 + l_2) + M_B \, l_2 = -6 (A_1 x_1 + A_2 x_2)$$

A I_1 I_1 I_2 l_1 l_2
I_2

$$2M_A\left(\frac{5}{I}\right) + M_B\left(\frac{5}{I}\right) = -6\left(\frac{180 \times 2.33}{5 \times I}\right)$$

$$10M_A + 5M_B = -503.28 \qquad \text{(i)}$$

Apply three moment theorem for the spans ABC

Properties of the simple beam *BMD*

$A_1 = 180 \text{ kNm}^2$	$A_2 = \frac{2}{3} \times 6 \times 45 = 180 \text{ kNm}^2$
$x_1 = \frac{5+3}{3} = 2.67 \text{ m}$	$x_2 = 3\text{m}$
$l_1 = 5\text{m}$	$l_2 = 6\text{m}$

. 5 Σ . 5 6 Σ . 6 Σ . 180 ✖ 2.67 180 ✖ 3 Σ

M_A I $+$ $2M_B$ $I +$ $1.5I$ $+$ M_C 1.5 I $=$ -6 5 $+ 6 \times 1.5$

$$5M_A + 18M_B + 4M_C = -6\,(96.12 + 60)$$

$$5M_A + 18M_B + 4M_C = -936.72 \qquad \text{(ii)}$$

Applying three moment theorem BCC'

As the end C is fixed imagine a span CC' of zero length and C' is simply supported

$. \; 6 \quad \Sigma \qquad . \; 6 \quad \Sigma \qquad . \; 180 \times 3 \quad \Sigma$

$M_B \quad 1.5I \quad + \qquad 1.5 \; + \; 0 \; + \quad _C \qquad = \qquad 6 \times 1.5 + 0$

$2M_C \qquad I \quad M\mathbf{j} \qquad -6$

31.62 kNm 60 kN 37.4 kNm

A *V* 3 m *D* 2 m *B* *V*

FIG. 11.17d

$$\sum V = 0; \quad V_{AB} + V_{BA} = 60 \tag{i}$$

$$\sum M_B = 0; \quad 5V_{AB} - 60(2) - 31.62 + 37.4 = 0 \tag{ii}$$

$$V_{AB} = 22.84 \text{ kN}$$

$$V_{BA} = 37.16 \text{ kN}$$

Span BC

$$M_D = 22.84(3) - 31.62 = 36.9 \text{ kNm}$$

37.4 kNm

10 kN/m

26.29 kNm

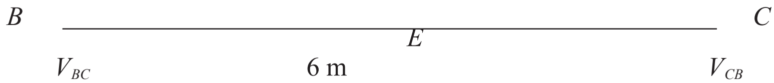


FIG. 11.17e

$$\sum V = 0; \quad V_{BC} + V_{CB} = 60 \qquad \text{(iii)}$$

M $\quad 6_2$

$$\sum M_C = 0; \quad -37.4 + 26.29 + 6V_{BC} - 10 \times 2 = 0$$

3_2

$$M_E = 31.85(3) - 37.4 - 10 \times 2 = 13.15 \text{ kNm}$$

31.85

22.84

+

+ F

A D − B − C

37.16

28.15 kN

2.81
5

FIG. 11.17f Shear force diagram

The location of zero shear in span *CB* is obtained by equating the shear force equation to zero as

$$(SF)_{xx} = 28.15 - 10x = 0$$

$$x = 2.815 \text{ m}$$

$$M_F = 28.15(2.815) - 10(2.815)^2/2 - 26.29$$

$$= 13.2 \text{ kNm}$$

+ 37.49 37.4 + 13.15

31.62 26.29 kNm

– –

A *D* *B* *F* *C*

FIG. 11.17g Bending moment diagram

EXAMPLE 11.18: A continuous beam *ABCD* is of uniform section as shown in figure. *EI* is constant. Draw the *SFD* and *BMD*

10 kN/m

6 m C 6 m D

A 6 m B

E *F* *G*

FIG. 11.18a

Solution

The simple beam moments are

$$M_E = M_F = M_G = \frac{10 \times 6^2}{8} = 45 \text{ kNm}$$

45 45 45 kNm

A *B* *C* *D*

FIG. 11.18b Simple beam BMD

M_B M_C

A *D*

B *C*

FIG. 11.18c Pure moment diagram

Considering spans ABC

$$A_1 = \frac{2}{3} \times 6 \times 45 = 180 \text{ KNm}^2$$

$$x_1 = 3\text{m}$$

$$+ 2M_B(6 + 6) + 6M_C = -6\left[\frac{180 \times 3}{6} + \frac{180 \times 3}{6}\right]$$

$$24M_B + 6M_C = -6(90 + 90) = -1080 \tag{1}$$

Considering span BCD

$$6M_B + 2M_C(6 + 6) + 6M_D = -6 \left(\frac{180 \times 3}{6} + \frac{180 \times 3}{6} \right)$$


10 kN/m
36 kNm
A
B

[redacted]

$$\sum \;{}_{B} = 0; \quad 6V_{AB} + 36 - 10 \times 2 = 0 \qquad \text{(ii)}$$

$\therefore$

Consider span BC

10 kN/m

36 kNm

36 kNm

B *V*

6 m

V *C*

BC CB

FIG. 11.18e

$$\sum V = 0; \quad V_{BC} + V_{CB} = 60 \tag{iii}$$

$$\sum M_C = 0; \quad 6V_{BC} - 36 + 36 - 10 \times \frac{6^2}{2} = 0 \tag{iv}$$

$$V_{BC} = 30 \text{ kN}, \; V_{CB} = 30 \text{ kN}$$

Span CD

36 kNm

10 kN/m

[redacted]

$\sum D = 0; \quad 6V_{CD} - 36 - 10 \times 2 = 0$ 2.4

FIG. 11.18g Shear force diagram

The location of zero shear is calculated as

$$24 - 10x_1 = 0$$

$$x_1 = 2.4\text{m}$$

$$M_E = 24(2.4) - 10(2.4)^2/2 = 28.8 \text{ kNm}$$

$$M_F = 30(3) - 36 - 10 \times 3^2/2 = 9.0 \text{ kNm}$$

$$M_G = 24\ (2.4) - 10(2.4)^2/2 = 28.8 \text{ kNm}$$

28.8 36 + 9 36 28.8 kNm

+ – – +

A *E* *B* *F* *C* *G* *D*

FIG. 11.18h Bending moment diagram

EXAMPLE 11.19: Analyse the continuous beam by three moment theorem. Also draw SFD and BMD.

A *E* *B* *F* *C* *G* *D*

FIG. 11.19a

49

A *E* *B* *F* *C* *G* *D*

FIG. 11.19b Simply supported BMD

M_B M_C

A *B* *C* *D*

FIG. 11.19c Pure moment diagram

SOLUTION

Properties of the simple beam *BMD*

$A_1 = 2 \times 3 \times 28.1 = 56.2$
kNm^2
3

$x_1 = 1.5\text{m}$

$l_1 = 3\text{m}$

$A_2 = 2 \times 2.8 \times 49 = 91.47$
kNm^2
3

$x_2 = 1.4\text{m}$

$l_2 = 2.8\text{m}$

$A_3 = 80\ \text{kNm}^2$

$x_3 = 2\text{m}$

$l_3 = 4\text{m}$

Applying three moment theorem for spans ABC

$$M_A(3) + 2M_B(3 + 2.8) + 2.8M_C = -6\left(\frac{56.2 \times 1.5}{3} + \frac{91.47 \times 1.4}{2.8}\right)$$

$$11.6M_B + 2.8M_C = -6(28.1 + 45.74)$$

$$11.6M_B + 2.8M_C = -443 \tag{i}$$

Applying three moment theorem for spans BCD

$$2.8M_B + 2M_C(2.8 + 4) + 4M_D = -6 \quad . \; \frac{91.47 \times 1.4}{2.8} \quad \frac{80 \times 2}{4} \; \Sigma$$

+

$$2.8M_B + 13.6M_C = -6\,(45.74 + 40)$$

$$2.8M_B + 13.6M_C = -514.44 \qquad \text{(ii)}$$

Solving (i) and (ii)

Free body diagrams of *AB*, *BC* and *CD*

30.58 kNm 50 kN/m 31.53 kNm 31.53 kNm 15 kN/m

25 kN/m 30.58 kNm

V_{B2} 2.8 m V_{C1} V_{C1} 4.0 m V_D

V_A 3 m V_{B1}

FIG. 11.19d

FIG. 11.19e

FIG. 11.19f

Static equilibrium of spans AB, BC and CD

$\sum V = 0;$	$\sum V = 0;$	$\sum V = 0;$
$V_A + V_{B1} = 75$ (i)	$V_{B2} + V_{C1} = 140$ (iii)	$V_{C2} + V_D = 60$ (v)
$\sum M_B = 0;$	$\sum M_C = 0;$	$\sum M_D = 0;$

$$3V_A + 30.58 \quad \frac{25}{2 \times 3^{2}} = 0 \quad \text{(ii)}$$

$-$

$31.53 - 30.58 +$

$2.8V_{B2}$

2.8^2

4_2

$$-31.53 + 4V_{C2} - 15 \times 2 = 0 \quad \text{(vi)}$$

$$-50 \times \quad 2 = 0 \quad \text{(iv)}$$

$V_A = 27.3$ kN

$V_{B1} = 47.7$ kN

$V_{B2} = 69.66$ kN

$V_C = 70.34$ kN

$V_{C2} = 151.53/4 = 37.88$ kN

$V_D = 22.12$ kN

27.3 69.66 37.88

+ + +

x_1 − x_2 − x_3 −

47.7 kN 70.34 22.12

FIG. 11.19g Shear force diagram

The locations of shear forces in zones *AB*, *BC* and *CD* are

$27.3 - 25x_1 = 0$	$69.66 - 50x_2 = 0$	$37.88 - 15x_3 = 0$
$x_1 = 1.09$ m	$x_2 = 1.39$ m	$x_3 = 2.52$ m

14.9 30.58 + 17.94 31.53 + 16.3
28.1 + 30 kNm

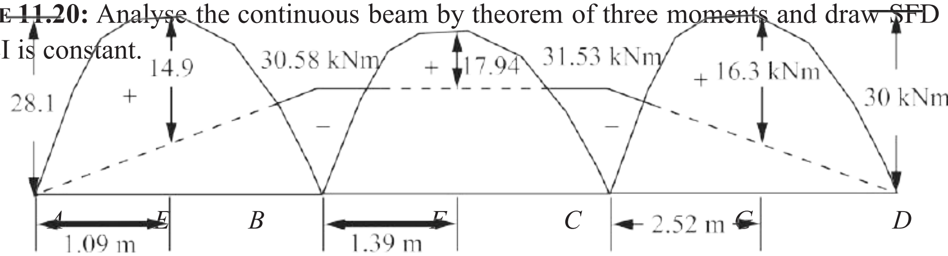
$$M_1 = 27.3(1.09) - 25 \times 1.09^2/2 = 14.9 \text{ kNm}$$

1.09 1.39 2.52 m

$$M_2 = -30.58 + 69.66(1.39) - 50 \times 1.39^2/2 = 17.94 \text{ kNm}$$

$$M_3 = -31.53 + 37.88(2.52) - 15 \times 2.52^2/2 = 16.3 \text{ kNm}$$

FIG. 11.19h Bending moment diagram



EXAMPLE 11.20: Analyse the continuous beam by theorem of three moments and draw SFD and BMD. EI is constant.

FIG. 11.20a

A *B* *C*

FIG. 11.20b Simple beam BMD for span *ABC*

B *C* *D*

FIG. 11.20c Simple beam BMD for span *BCD*

FIG. 11.20d Pure moment diagram

SOLUTION

Referring to Fig. 11.20 b

Properties of simple beam *BMD*

$$A_1 = \frac{1}{2} \times 10 \times 4 = 20 \text{ kNm}^2$$

$$x_1 = 2\text{m}$$

$$l_1 = 4\text{m}$$

$$A_2 = \frac{2}{3} \times 6 \times 22.5 = 90 \text{ kNm}^2$$

$$x_2 = 3\text{m}$$

$$l_2 = 6\text{m}$$

Applying three moment theorem for spans ABC,

$. 20 \times 2 \quad 90 \times 3\ \Sigma$

$4M_A + 2M_B(4 + 6) + 6M_C = \quad 4 \quad 6$

-6

$+$

$$20M_B + 6M_C = -6(10 + 45)$$

$$20M_B + 6M_C = -330 \quad \text{(i)}$$

Referring to Fig. 11.20 c

Properties of simple beam *BMD*

$$A_1 = \frac{2}{3} \times 6 \times 22.5 = 90 \text{ kNm}^2$$

$$x_1 = 3\text{m}$$

$$l_1 = 6\text{m}$$

$$A_2 = \frac{1}{2} \times 5 \times 18 = 45 \text{ kNm}^2$$

$$x_2 = \frac{5 + 2}{3} = 2.33 \text{ m}$$

$$l_2 = 5\text{m}$$

Applying three moment theorem for spans BCD
Considering span BCD

$$6M_B + 2M_C(6 + 5) + 5M_D = -6\left[\frac{90 \times 3}{6} + \frac{45 \times 2.33}{5}\right]$$

$$6M_B + 22M_C = -6(45 + 20.97)$$

$$6M_B + 22M_C = -395.82 \qquad \text{(ii)}$$

Solving (i) and (ii)

$$M_B = 12.09 \text{ kNm.}$$

$$M_C = 14.69 \text{ kNm.}$$

Shear force and bending moment values for spans *AB, BC* and *CD*.

.09 kNm

A

V_{AB} V_{BA}

FIG. 11.20e

$$\sum V = 0; \quad V_{AB} + V_{BA} = 10 \tag{i}$$

$$\sum M_B = 0; \quad 4V_{AB} + 12.09 - 10(2) = 0 \tag{ii}$$

Solving (i) and (ii)

$$M_E = 1.98(2) = 3.96 \text{ kNm}$$

Span BC

12.09 kNm

5 kN/m

14.69 kNm

B V_{BC} 6 m C V_{CB}

FIG. 11.20f

$$\sum V = 0; \quad V_{BC} + V_{CB} = 6(5) = 30 \text{ kN} \qquad \text{(iii)}$$

$$\sum M_B = 0; \quad 6V_{BC} + 14.69 - 12.09 \; \frac{5 \times 6^2}{2} \; = 0$$

$-$

The location of shear force is zero is found out as

$$14.56 - 5x = 0$$

$$x = 2.91 \text{ m}$$

Hence We

Max BM 14 56 2 91 12 09 5 2.91^2 9 11 kNm

$+ \quad = \; . \; (\; . \;) \quad . \; - \times \quad 2 \; = \; .$

Span CD

15 kN

14.69 kNm

[redacted]

1.98 14.56 8.94

+ + +

A E B C D

− − −

8.02 14.69 6.06 kN

FIG. 11.20h Shear force diagram

+

3.96 + 12.09 − 9.11 15.44 kNm − + 12.12

A E B C D

FIG. 11.20i Bending moment diagram

EXAMPLE 11.21: A continuous beam *ABCD* is simply supported at *A* and *D*. It is continuous over supports *B* and *C*. $AB = BC = CD = 4$ m. EI is constant. It is subjected to uniformly distributed load of 8 kN/m over the span *BC*. Draw the shear force diagram and bending moment diagram.

8 kN/m

A *B* *E* *C* *D*

FIG. 11.21a

SOLUTION

The simple beam moment

$$M_E = \frac{8 \times 4^2}{8} = 16 \text{ kNm}$$

16 kNm

A *B* *E* *C* *D*

FIG. 11.21b Simple beam bending moment diagram

M_B M_C

A *B* *C* *D*

FIG. 11.21c Pure moment diagram

Consider span ABC

Applying three moment theorem;

$$4M_A + 2M_B(4 + 4) + 4M_C = -6 \left[0 + 2 \times \frac{4 \times 16 \times 2}{4} \times 3 \right]$$

$$16M_B + 4M_C = -128 \qquad \text{(i)}$$

Consider span BCD

$$4M_B + 2M_C(4 + 4) + 4M_D = -6\left[\frac{\frac{2}{3} \times 4 \times 16 \times 2}{4} + 0\right]$$

$$4M_B + 16M_C = -128 \qquad \text{(ii)}$$

6.4 kNm 8 kN/m 6.4 kNm

B V_{BC} 4 m *C* V_{CB}

FIG. 11.21e

$$\sum V = 0;\ V_{BC} + V_{CB} = 8(4) = 32 \qquad \text{(iii)}$$

$$\sum M_C = 0; \quad -6.4 + 6.4 + 4V_{BC} \quad \frac{8 \times 4^2}{2} \quad = 0 \qquad \text{(iv)}$$

$-$

span CD

6.4 kNm

C *D*

V_{CD} V_{CD}

FIG. 11.21f

[redacted]

B

$\sum M_D = 0; \quad -6.4 + 4V_{CD} = 0$ (vi)

A – C D

–1.6

EXAMPLE 11.22: Analyse the beam shown in figure by SFD and BMD. EI is constant.

16

10 kN/m

FIG. 11.21g Shear force diagram

16 kNm

\+

6.4 6.4

– –

A B C D

FIG. 11.21h Bending moment diagram

A 6 m *B* 6 m *C* 6 m *D* 6m *E*

FIG. 11.22a

45 45

A *B* *C* *D* *E*

FIG. 11.22b Simple beam BMD

M_B M_C M_D

A B C D E

FIG. 11.22c Pure moment diagram

Properties of simple beam

BMD

$$A_1 = \frac{2}{3} \times 6 \times 45 = 180 \qquad A_2 = \frac{2}{3} \times 6 \times 45 = 180 \text{ KNm}^2$$

$$x_1 = 3\text{m} \qquad x_2 = 3\text{m}$$

$$l_1 = 6\text{m} \qquad l_2 = 6\text{m}$$

Applying 3 moment theorem for the spans ABC

180 ✖ 3 Σ

. 6

$6M_A + 2M_B(6 + 6) + 6M_C = -6\ 0$

+

$$24M_B + 6M_C = -540 \quad \text{(i)}$$

Applying 3 moment theorem for the spans BCD

$$6M_B + 2M_C(6 + 6) + 6M_D = -6\left(\frac{180 \times 3}{6} + \frac{180 \times 3}{6}\right)$$

$$6M_B + 24M_C + 6M_D = -6(180) = -1080$$

$$M_B + 4M_C + M_D = -180 \qquad \text{(ii)}$$

Applying 3 moment theorem for spans CDE

. 180 ✖ 3 Σ

$$6M_C + 2M_D(6 + 6) + 6M_E = \quad 6 \quad + 0$$
$$-6$$

FIG. 11.22d

[redacted]

$$\sum M_B = 0; \quad 6V_{AB} + 12.86 = 0 \qquad \text{(ii)}$$

span BC

12.86 kNm

10 kN/m

38.57 kNm

B C

V_{BC} 6 m V_{CB}

FIG. 11.22e

$$\sum V = 0; \quad V_{BC} + V_{CB} = 60 \qquad \text{(iii)}$$

$$\sum M_C = 0; \quad -12.86 + 38.57 + 6V_{BC} - \frac{10 \times 6^2}{2} = 0 \qquad \text{(iv)}$$

span CD

38.57 kNm

10 kN/m

12.86 kNm

C

D

6 m

V_{CD}

V_{DC}

[redacted]

12.86 kNm

V_{DE} 6 m V_{ED}

FIG. 11.22g

Solution

$$M_B = -20(1) = -20 \text{ kNm}$$

The simple beam moments are

$$M_F = \frac{20 \times 4^2}{8} = 40 \text{ kNm}$$

$$M_G = \frac{60 \times 4}{4} = 60 \text{ kNm}$$

40 kNm 60 kNm

B *C* *F* *D* *G* *E*

FIG. 11.23b (b) Simple beam BMD

M

20

B *C* *D* *E*

FIG. 11.23c (c) Pure moment diagram

Apply 3 moment theorem for the spans BCD

. 2 2 Σ

$$-20(3) + 2M_C(3 + 4) + M_D(4) = 0 + 3 \times 4 \times 40 \times 4$$

-6

$$-60 + 14M_C + 4M_D = -320$$

$$14M_C + 4M_D = -260 \qquad \text{(i)}$$

Apply 3 moment theorem for the spans CDE

. 2 2 1 2 Σ

$M_C(4) + 2M_D(4 + 4) + 4M_E = \quad 3 \times 4 \times 40 \times 4 + 2 \times 4 \times 60 \times 4$
-6

$$4M_C + 16M_D = -6(53.33 + 60) = -680 \qquad \text{(ii)}$$

Solving (i) and (ii)

$$+ 6EI \left(\frac{\delta_A}{L_1} + \frac{\delta_C}{L_2} \right)$$

$$A_1x_1 = 960$$

$$A_2x_2 = \frac{2}{3} \times 6 \times 180 \times 3 = 2160$$

Σ 960 2160 Σ . 240 120 Σ

Substituting, $M_A \times 4 + 2M_B(4 + 6) = -6$ $\quad 4 + \quad 6 \quad + \; 6EI \quad EI \times 4 + EI \times 6$

$$4M_A + 20M_B = -3120 \quad \rightarrow \quad M_A + 5M_B = -780 \tag{2}$$

Solving (1) and (2), $M_A = -163.33$ kNm hogging BM.

$M_B = -123.33$ kNm

240 kN

40 kN/m

163.33 kNm 2 m

2 m

123.33 kNm

A *B* *B* 6 m *C* *O*

130 kN 110 kN 140.56 kN 94.44 kN

FIG. 11.25d Free body diagram of spans *AB* and *BC*

130 kN

140.56

A *B* *C* 94.44

FIG. 11.25e Shear force diagram

240

163.33 kNm +

123.33

− + 180

−

FIG. 11.25f Bending moment diagram

B *C*

Ans:

$$R_A = 45\text{kN}, \quad R_B = 165.5\text{kN}, \quad R_C = 69.5\text{kN}$$

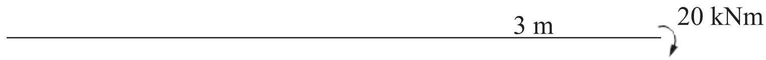
(11.3) A continuous beam of uniform section *ABCD* is supported and loaded as shown in □gure. If the support *B* sinks by 10 mm, determine the resultants and moments at the supports.

(11.3) A continuous beam of uniform section *ABCD* is supported and loaded as shown in Figure. If the support *B* sinks by 10 mm, determine the resultants and questions at the supports.

Assume $E = 2(10)^5$ N/mm^2; $I = 6(10)^7$ mm^4

40 kN

10 kN/m


3 m
20 kNm

A 4 m *B* 6 m *C* 1m *D*

$$\boxed{EI = Constant}$$

Ans:

$V_{AB} = +16.5$ kN, $V_{BA} = +23.5$, $V_{BC} = +19$, $V_{CB} = +21.0$

$M_B = -14$ kNm

(11.4) Determine the reactions at A, B and C of the continuous beam shown in figure.

8 kN

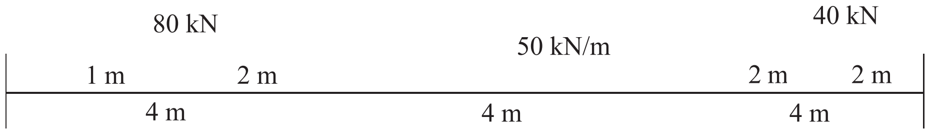
3 kN/m


1 m
3 m
A
4 m
5 m
1

Ans:

$$V_{AB} = 6.75 \text{ kN}, \quad V_{BA} = 1.25, \quad V_{BC} = 6.31; \quad V_{CB} = 8.69$$
$$M_A = -3.31 \text{ kNm}, \quad M_B = -3.87, \quad M_{CB} = +7.44$$

(11.5) Analyse the continuous beam shown in Figure and determine the reactions


80 kN
40 kN
50 kN/m
1 m
2 m
2 m
2 m
4 m
4 m
4 m

A B 2I C D

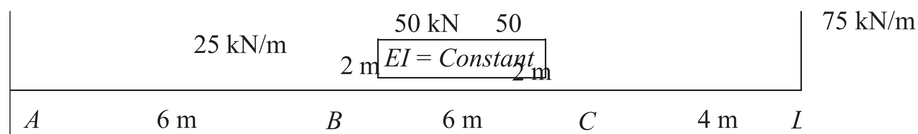
Ans:

$$V_{AB} = 41.68 \text{ kN}, V_{BA} = 38.32, \ V_{BC} = 102.88, V_{CB} = 97.12$$

$$V_{CD} = 28.03, \quad V_{DC} = 11.97, \quad M_A = -17.98 \text{ kNm},$$

$$M_B = -52.93, \quad M_C = -41.42, M_D = -9.29 \text{ kNm}$$

(11.6) Analyse three span continuous beam by three moment theorems. Draw the BMD and shear force diagram. Determine the end moments and reactions *EI* is constant.



Ans:

(i) $R_A = 75.39$ kN, $\quad R_B = 127.59$ kN, $\quad R_C = 97.85$ kN, $\quad R_D = 99.17$ kN

$M_A = -75.78$ kNm, $\quad M_B = -73.44$ kNm, $\quad M_C = -55.55$ kNm, $\quad M_d = -55.2$ kNm.

(11.7) Analyse and draw BMD and SFD for the beam shown in Figure. The values of second moment area of each span are indicated along the members. Modulus of elasticity is constant.

100 kN

30 kN/m

80 kN

40 kN

2.5 m 2.5 m 1.25 m 2.5 m 1.25 m

2I 3I 4I

A B 6 m C D

Ans:

$$M_A = -56.02 \text{ kNm}, \quad M_B = -75.47 \text{ kNm}, \quad M_C = -94.3 \text{ kNm}, \quad M_D = 0$$

(11.11) Determine the reactions and the support moment at *B*. Using Clapeyron's three moment theorem.

A *B* *C*

EI = *Constant*

Ans:

$$V_A = 4.81 \text{ kN}, \quad V_B = 0.31, \quad V_C = 4.88 \text{ kN}, M_B = -0.72 \text{ kNm}$$

(11.12) Analyse the continuous beam by three moment theorem, determine the support moments. No loads on span *AB*.

0.5 kN

2.5 kN

Ans:

$$M_A = -1.09 \text{ kNm}, \quad M_B = -2.188 \text{ kNm}, \quad M_C = -7.5 \text{ kNm}$$

UNIT - VI

CONTINUOUS BEAMS - SLOPE DEFLECTION METHOD

3.1 Introduction:-

The methods of three moment equation, and consistent deformation method are represent the FORCE METHOD of structural analysis, The slope deflection method use displacements as unknowns, hence this method is the displacement method.

In this method, if the slopes at the ends and the relative displacement of the ends are known, the end moment can be found in terms of slopes, deflection, stiffness and length of the members.

In- the slope-deflection method the rotations of the joints are treated as unknowns. For any one member bounded by two joints the end moments can be expressed in terms of rotations. In this method all joints are considered rigid; i.e the angle between members at the joints are considered not-to change in value as loads are applied, as shown in fig 1.

joint conditions:- to get θ_B & θ_C $M_{BA}+M_{BC}+M_{BD} = 0$ .... (1)

$M_{CB}+M_{CE} = 0$ .... (2)

Figure (1)

3.2 ASSUMPTIONS IN THE SLOPE DEFLECTION METHOD

This method is based on the following simplified assumptions.

1- All the joints of the frame are rigid, i.e, the angle between the members at the joints do not change, when the members of frame are loaded.

2- Distortion, due to axial and shear stresses, being very small, are neglected.

3.2.1 Degree of freedom:-

The number of joints rotation and independent joint translation in a structure is called the degrees of freedom. Two types for degrees of freedom.

In rotation:-

For beam or frame is equal to D_r.

$$\mathbf{D_r = j\text{-}f}$$

Where:

D_r = degree of freedom.

j = no. of joints including supports.

F = no. of fixed support.

<u>In translation:-</u>

For frame is equal to the number of independent joint translation which can be give in a frame. Each joint has two joint translation, the total number or possible joint translation = 2j. Since on other hand each fixed or hinged support prevents two of these translations, and each roller or connecting member prevent one these translations, the total number of the available translational restraints is;

$2f + 2h + r + m$ where

f = no. of fixed supports.

h = no. of hinged supports.

r = no. of roller supports.

m = no. of supports.

The degree of freedom in translation, D_t, is given by:-

$$\mathbf{D_t = 2j-(2f+2h+r+m)}$$

The combined degree of freedom for frame is:-

$$D = D_r + D_t$$

$$= j - f + 2j - (2f + 2h + r + m)$$

$$D = 3j - 3j - 2h - r - m$$

The slop defection method is applicable for beams and frames. It is useful for the analysis of highly statically indeterminate structures which have a low degree of kinematical indeterminacy. For example the frame shown in fig. 2.a

The frame (a) is nine times statically indeterminate. On other hand only tow unknown rotations, θ_b and θ_c i.e Kinematically

indeterminate to second degree- if the slope deflection is used.
The frame (b) is once indeterminate.

3.3 <u>Sign Conventions:-</u>

Joint rotation & Fixed and moments are considered positive when occurring in a clockwise direction.

$$\theta_{A1} = \frac{2}{3}\frac{M_A.L}{2\,EI} = \frac{M_A.L}{3\,EI}$$

$$\theta \quad = \underline{1} = \quad \underline{MA.L} = \quad -MA.L$$

BI 3 2 EI 6 EI

hence $\theta B1 = \frac{1}{2} \theta A1$

$\theta A2 = \underline{1} \quad \underline{MB.L} = \underline{-MB.L}$

3   2 EI   6 EI

θ   = 2 MB.L = MB.L

B2 3 2 EI 3 EI - -

$$\theta_{B1} + \theta_{B2} = 0$$

Hence: $M_A = 2M_B$

and $\theta_A = \theta_{A1} - \theta_{A2}$

–

$$= \frac{M_A . L}{3\ EL} \qquad \frac{M_A . L}{12\ EL}$$

$$\theta_A = \frac{3MA.L}{12EI}$$

$MA = \frac{4EI}{L}.\theta A$
$MB = \frac{2EI}{L}.\theta A$

Relation between Δ & M

$R = \frac{\Delta}{L}$

by moment area method or
by conjugate beammethod.

$\Delta = \sum M_{at\ B}$

$$= \frac{M.L}{4EI} \left(\frac{2L}{3} \right)$$

$$= \frac{M.L^2}{6\,EI}$$

$6EI$

- –

$$M = \quad \Delta$$

$$L_2$$

$$= \frac{6EI}{L}.R$$

R (+ ve) when the rotation of member AB with clockwise.

3.4 **Fixed and moments:**

As given in the chapter of Moment distribution method.

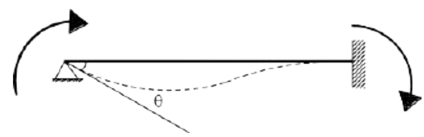
3.5 Derivation of slope deflection equation:-

$M_{a1} = \frac{4EI}{L}\theta_A$

$M_{b1} = \frac{2EI}{L}\theta_A$

$M_{a2} = \frac{2EI}{L}\theta_B$

$M_{b2} = \frac{4EI}{L}\theta_B$



Required M_{ab} & M_{ba} in term of (1) θ_A, θ_B at joint

(2) rotation of member (R)

(3) loads acting on member

First assume:-

Get Mf_{ab} & Mf_{ba} due to acting loads. These fixed and moment must be corrected to allow for the end rotations θ_A,θ_B and the member rotation R.

The effect of these rotations will be found separately.

M_{a} $= \frac{4EI}{L} . \theta_{A}$

1

2EI

$$M_{b1} = \frac{}{L} . \theta_A$$

$$M_{b2} = \frac{4EI}{L} . \theta_B$$

$$M_{a2} = \frac{2EI}{L} . \theta_B$$

$$M_{b3} = M_{a3} = -6\,EI \,.\Delta\, L_2$$

$$= -\frac{6\,EI}{L}.R$$

by Superposition;

$$M_{ab} = Mf_{ab} + M_{a1} + M_{a2} + M_{a3}$$

$$Mf_{ab} + \frac{4EI}{L}.\theta_A + \frac{2EI}{L}\theta_B + -\frac{6EI}{L}.R$$

In case of relative displacement between the ends of members, equal to zero (R = 0)

$$M_{ab} = Mf_{ab} + \frac{2EI}{L}(2\theta_a + \theta_b)$$

$$M_{ba} = Mf_{ba} + \frac{2EI}{L}(2\theta_b + \theta_A)$$

2 EI

The term (L   ) represents the relative stiffness of member say (K) <u>hence:</u>

$$M_{ab} = Mf_{ab} + K_{ab}\,(2\theta_A + \theta_b)$$

$$M_{ba} = Mf_{ba} + K_{ba}(2\theta_B + \theta_a)$$

Note:

$\frac{\Delta}{L} = R$ is (+ ve) If the rotation of member with clockwise.

And (– ve) If anti clockwise.

$$M = -\frac{6\,EI}{L^2}.\Delta \quad \text{(with + ve R)}$$

$$M = -\frac{6\,EI}{L^2}.\Delta \quad \text{(with − ve R)}$$

3-5-1 **Example 1**

Draw B.M.D. S.F. D

Solution:-

1- Relative stiffness:- $K_{AB} : K_{BC}$ = 1 : 2.66 1 : 2

$6 \qquad 8$

2- Fixed and Moment:-

$$MF_{BA} = \frac{3 \times 6^2}{12} = -9\ t.m.$$

3×6^2 $\qquad$ 3×8^2

$$MF_{BA} = + \quad \frac{12}{12} \quad = + 9 \quad , \qquad MF_{BC} = + \qquad\qquad = - 18$$

$$MF_{CB} = +\frac{3 \times 8^2}{12} = +18$$

3- Two unknown $\theta_B + \theta_C$ then two static equations are required. 1) $\sum M_B = 0$

2) $M_C = 0$

<u>Hence:</u>

$$M_{BA} + M_{BC} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

M_{BC} = 0………………. (2)

But:

M_{AB} = - 9 + θ_B

M_{BA} = 9 + 1 ($2\theta_B$)

M_{BC} = -16 + 2 ($2\theta_B + \theta_C$)

M_{CB} = +16 + 2 ($2\theta_C + \theta_B$)

From eqns. (1&2)

$9 + 2\theta_B + (- 16 + 2 (2\theta_B + \theta_C) = 0$

$6\theta_B + 2\theta_C = 7$ …….(3)

and $4\theta_C + 2\theta_B = - 16$

$2\theta_C + \theta_B = - 8$.................. (4)

from 3 & 4

$5\ \theta_B = 15$

$\theta_B = \frac{15}{5} = 3.0$

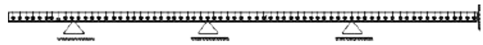
$\theta_C = - 5.5$

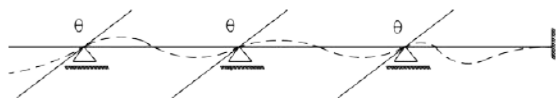
1.e. M_{AB} = - 9 + 3.4 = 5.6 t.m

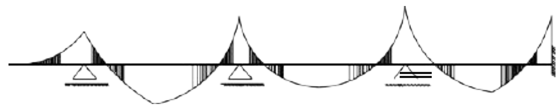
M_{BA} = 9 + 2 $\times$ 3.4 = 15.8 t.m

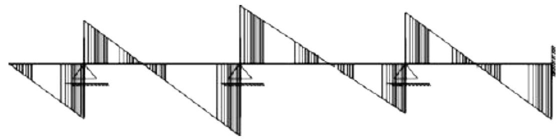
M_{BC} = - 18 + 2 (2 $\times$ 3.4) + (- 5.5) = - 15.0 t.m

M_{CB} = 16 + 2 (2.3 – 5.7 + 3.4) = 0.0 (0.k)




θ
θ
θ





1- **Unknowns** θ_A ، θ_B ، & θ_C

2- Fixed end Moment

$$MF_{AB} = MF_{BC} = MF_{CD} = \frac{2 \times 6^2}{12} = -6 \text{ t.m} \dots \text{etc}$$

3- Condition eqns.

$M_{AB} = -4$ t.m, $M_{BA} + M_{BC} = 0$, & $M_{CB} + M_{CD} = 0$

4- Slope deflection equations

$M_{AB} = -6 + (2\theta_A + \theta_B) = -4$

$2\theta_A + \theta_B = 2$ .... (1)

$M_{BA} + M_{BC} = 0$

$+6 + (2\theta_B + \theta_A) - 6 + (2\theta_B + \theta_C) = 0$

$4\theta_B + \theta_A + \theta_C = 0$ .... (2)

$M_{CB} + M_{CD} = 0$

$= 6 + 2\theta_C + \theta_B - 6 + 2\theta_C = 0$

$4\theta_C + \theta_B = 0$ .... (3)

From eqn.3 $\theta_C = -\frac{\theta B}{4}$

Substitute in eqn. (2)

Hence: $3.75\ \theta_B + \theta_A = 0$ .... (2)

$0.5\theta_B + \theta_A = 1$ .... (2)

$3.25\theta_B = -1$

$\theta_B = -1$

$\theta_A = 1.15$

$\theta_C = 0.077$

Hence:

M_{AB}	= - 6 + 2xl.15 + (- .307)	
	= - 4 t.m 0.K	
M_{BA}	= 6 + 2x (- .307) + 1.15	= 6.536 t.m
M_{CB}	= 6 + 2x .77 + (- .307)	= 5.85 t.m
M_{DC}	= 6 + .077	= 6.077 t.m

Solution:-

1- Unknown displacements are θ_B & θ_D

2- Equations of equilibrium are:-

$M_{DB} = 0$.. (1)

$M_{BA} + M_{BD} + M_{BC} = 0$... (2)

3- Relative Stiffness:-

K_{AB}: K_{BC}: K_{BD} = 35:31. 5:22 ; 51. 56:1. 4:1.0.

4- Fixed and Moments:

$$MF_{AB} = -\frac{9\times6\times3\times3}{9\times9} = -6 \quad t.m$$

$$MF_{BA} = \frac{9\times 6\times 3\times 6}{9\times 9} = 12 \; t.m$$

$$-\,3\times 7^2$$

$$MF_{BD} = 12 = -12.25 \quad t.m$$

$$MF_{DB} = \frac{-3 \times 7^2}{12} = -12.25 \quad t.m$$

From the equations 1 & 2 hence;

$M_{DB} = MF_{DB} + (2\theta_D + \theta_B)$

$= 12.25 + 1\ (2\theta_D + \theta_B) \quad = 0$

$2\theta_D + \theta_B + 12.25 \quad = 0$---------(3)

and $M_{BA} \quad = 12 + 1.56\ (2\theta_B)$

$M_{BD} \quad = 12.25 + 1.0\ (2\theta_B + \theta_D)$

$M_{BC} \quad = 0 + 1.4 \quad (2\theta_B + 0)$

i.e.

$12+1.56\ (2\theta_B) - 12.25 + 2\theta_B + \theta_D + 1.4\ (2\theta_B) = o$

$7.92\theta_B + \theta_D - .25 \quad = 0$--------------- (4)

$\underline{0.5\theta_B + \theta_D + 6.125} \quad = 0$--------------- (3)

i.e $\quad 7.42\ \theta_B \quad - 6.375 \quad = o$

$\theta_B \quad = 0.86$

$\theta_D \quad = - 6.55$

<u>Hence:</u>

$M_{BA} = 12 + 1.56\ (2\times .86) = 14.68$ t.m

$M_{BD} = -12.25 + (2 \times .86 - 6.55 \times 1) \quad = -17.08$

$M_{BC} = 1.4\ (2 \times .86) \quad = 2.41$

$M_{DB} = 12.25 + (2 \times -6.55) \quad = \text{zero}$

$$M_{CB} = \frac{1}{2} MBC = 1.205$$

$$M_{AB} = -6 + 1.56\ (.86) \qquad = -\ 4.66$$

Two equilibrium eqns.

$M_{AB} + M_{AA} = 0$.. (1)

$M_{BB} + M_{BA} + 4 = 0$... (2)

Slope deflection eqns.

$M_{AB} = o + 1.6\ (2\theta_A + \theta_B)$

$$M_{AA} = \frac{-10 \times 16}{8} + (2\theta_A + \theta A)$$

$M_{AA} = - 20 + \theta_A$

$M_{BA} = o + 1.6 (2\theta_B + \theta_A)$

$M_{BB} = - 42.67 + (2\theta_B + \theta_B)$

$= - 42.67 + \theta_B$

Hence:

$3.2\theta_A + 1.6\theta_B + \theta_A - 20 = o$

$4.2\theta_A + 1.6\theta_B = 20$.............................. (1)

$- 42.67 + 4.2 \theta_B + 1.6\theta_A + 4 = 0$

$1.6\theta_A + 4.2\theta_B = 38.67$......................... (2)

$1.6\theta_A + 0.61\theta_B = 7.62$........................... (1)

$3.59\theta_B = 31.05$

$\theta_B = 8.65$

$\theta_A = 1.46$

$M_{AB} = -18.52$

M_{BA} = 30

M_{BB} = - 34

Example 5

Draw B.M.D for the shown frame

Solution:-

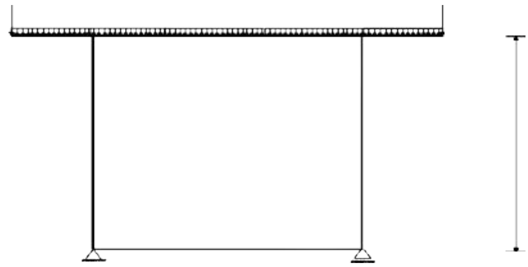
- **Two condition equations.**

$M_{AA} + M_{AB} = 0$...................................... (1)

$M_{BA} + M_{BB} + 8 = 0$...................................... (2)

- **Relative stiffness** $\frac{1}{16}$ **:** $\frac{1}{10}$ **= 1:1.6**

- Slope deflection equations:



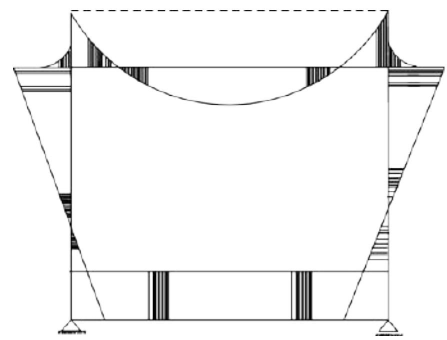
$M_{AA} = (2\theta_A - \theta_A) = \theta_A$

$M_{AB} = (2\theta_A - \theta_B) \times 1.6$

$M_{BA} = (2\theta_B - \theta_A) \times \theta_A$

-

$M_{BB} = 42.67 + (2\theta_B - \theta_B)$



Hence:

$\theta_A + 3.2\theta_A + 1.6V_B = 0$

$4.2\theta_A + 1.6\theta_B \qquad = 0 \ldots.\ (1)$

$3.2\theta_B + 1.6\theta_A + \theta_B - 42.67 + 8 = 0$

—

$$4.2\theta_B + 1.6\theta_A = 34.67 \ldots (2)$$

By Solving 1 & 2 $\theta_A = -3.68$, $\theta_B = 9.66$

Hence $M_{AA} = -3.68$, $M_{AB} = 3.68$ t.m

$M_{BA} = 25$ $M_{BB} = 33$

Example 6:

- Draw B.M.D for the given structure.

Solution:- once statically indeterminate.

1- Fixed end moments

$$MF_{AB} = -\frac{8 \times 20}{8} = -20 \text{ t.m}$$

$$MF_{BA} = -\frac{8 \times 20}{8} = -20 \text{ t.m}$$

$$MF_{BC} = -\frac{4 \times 10}{8} = -5 \text{ t.m}$$

$MF_{CB} = -\ 10 \times 8 = -10$ t.m 8

$MF_{DB} \qquad\qquad = 10$ t.m

2- <u>From Static:-</u> $\sum M_B = o$

$M_{BA} + M_{BC} + M_{BD} = 0$

$M_{BA} = MF_{BA} + (2\theta_B)$

$M_{BA} = 20 + 2\theta_B$ ……………….. (1)

$M_{BC} = - 5 + 2\theta_B$ ……………….. (2)

$M_{BD} = - 10 + 2\theta_B$ ………………. (3)

Hence: $5 + 6\theta_B = o$

$\theta_B = - o.833$

Hence:

M_{BA} = 18.34 t.m , MBC = -6.67, MBD = -11.67 t.m

$M_{AB} = - 20$ = - 20.833 t.m

$M_{CB} = 5 + \theta B$ = - 4.167 t.m

$M_{DB} = 10 + \theta B$ = 9.167 t.m

Example 7:

Draw B.M.D for the shown frame

Solution:

" 3 time statically ind." θ_A , θ_B , & θ_C

1- **Fixed end moments:**

$MF_{AB} = - 10$

$MF_{BA} = + 10$

$MF_{BC} = - 25$

$MF_{CD} = MF_{DC} =$ zero

2- **Relative Stiffness** 1:1:1

$M_{AB} = 0$.. (1)

$M_{BA} + M_{BC} = 0$.. (2)

$M_{CB} + M_{CD} = 0$.. (3)

Equs.

$$M_{AB} = -10 + (2\theta_A + \theta_B)$$

$$M_{BA} = 10 + (2\theta_B + \theta_A)$$

$$M_{BC} = -25 + 2\theta_B + \theta_C$$

$$M_{CB} = 25 + 2\theta_C + \theta_B$$

$$M_{CD} = 2\theta_C$$

$$M_{DC} = \theta_C$$

From 1,2 & 3

$2\theta_A + \theta_B = 10$ .... (1)

$4\theta_B + \theta_A + \theta_C = 15$ .... (2)

$4\theta_C + \theta_B = - 25$ .... (3)

By solving the three eqns. hence;

$\theta_A = 2.5$ $\theta_B = 5$ $\theta_C = - 7.5$

Substitute in eqns of moments hence;

$M_{AB} = - 10 + 5 = \text{zero (o.k)}$

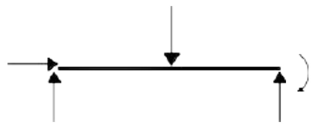
$M_{BA} = 10 + 10 + 2.5 = 22.5$ t.m

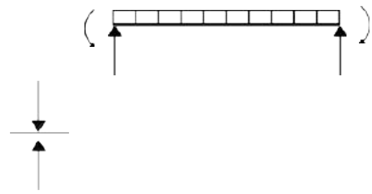
$M_{BC} = - 25 + 10 - 7.5 = - 22.5$ t.m

$M_{CB} = 25 - 15 + 5 = 15$ t.m

$M_{CD} = -15$ t.m

$M_{DC} = -7.5$ t.m





3-6 Frames with Translation

Examples to frames with a single degree of freedom in translation.

Example 8:

Draw B.M.D for the shown frame.

1- **Unknowns**: θB , θC , Δ

2- Relative stiffness

$K_{AB} : K_{BA} : K_{CD}$

$\frac{1}{4} : \frac{2}{8} : \frac{1.5}{6}$

1: 1 : 1

3- Fixed end moments

$MF_{AB} = o$ $M_{BA} = o$

$MF_{BC} = MCB = zero$

$MF_{CD} = - 6$ t.m

$MF_{DC} = + 6$ t.m

4- From Statics the equilibrium eqns

$M_{BA} + M_{BC} = 0$.................................... (1)

$M_{CB} + M_{CD} = 0$..................................... (2)

5- Shear equation (In direction of X, $\sum x = o$)

$6 + X_A + X_D - 8 \qquad = o$

$$6 + \frac{M_{BA} + M_{AB}}{4} + \frac{M_{CD} + M_{DC}}{6} - 4 = 0 \qquad (3)$$

$$hence \quad X_A = \frac{MBA + MAB}{4} \quad and \quad xD = \frac{MCD + MDC + 4}{6}$$

6- Slope deflection eqns:

$$M_{BA} - 0 + 1\ (2\theta_B - 3\ \frac{\Delta}{4}),\ M_{AB} = 0 + 1\ (\theta_B - 3.\ \frac{\Delta}{4})$$

$$M_{BC} = 0 + 1\,(2\theta_B + \theta_C)$$

<u>Hence:</u> $4\theta_B - 0.75\Delta + \theta_C = 0$ (1)

$$M_{CB} = 0 + 1\,(2\theta_C + \theta_B)$$

$$M_{CD} = -6 + 1\ (2\theta_C - 3\ \frac{\Delta}{6}),$$

$$M_{DC} = +6 + 1\left(\theta_C - 3\frac{\Delta}{4}\right)$$

Hence:

$$4\,\theta_C + \theta_B - \frac{1}{2}\Delta = 6 \qquad (2)$$

$\Delta = 0$

$2 + (2\theta_B - .75\ \Delta) + (1\theta_B - .75\ \Delta) + (-6 + 2\theta_C - \Delta) + (6$
$+ 1\theta_C -$

4 6

$$2 + 0.75\,\theta_B - .375\,\Delta + \frac{1}{2}\,\theta_C - 0.1667\Delta = o$$

$$\theta_B + .67\ \theta_C - 072\Delta = -\ 2.66 \quad (3)$$

Subtract (3) from (2)

$$3.33\ \theta\ \tfrac{1}{2} + 0.288\ \Delta = 8.33$$

$$\theta_C - 0.067\ \Delta = 2.6 \quad (4)$$

Subtract (1) from (2) × (4)

$$15\ \theta_C - 1.25\ \Delta = 24$$

$$\theta_C - 0.08\ \Delta = 1.6 \quad (5)$$

From (4) & (5) $0.147\Delta = 1$

$$\Delta = 6.80$$

$$\theta_C = 2.149$$

$$\theta_B = 0.799$$

$M_{BA} = -3.5$ t.m ,$M_{AB} = -4.301$ t,m, $M_{BC} = 3.79$

$M_{CB} = 5.1$ t.m ,$M_{CD} = -5.1$ t.m , $M_{DC} = 4.744$

Example 9:-

Write the shear & condition eqns for the following frame.

Solution:-

Three unknowns: θB , θC, Δ

Condition equations:

$M_{BA} + M_{BC} = o$ (1)

$M_{CB} + M_{CD} = o$ (2)

Shear eqn.

$X_A + X_B + P1 + P2 = o$

$$\left(-\frac{P1}{2}+\frac{M_{AB}+M_{BA}}{h1}\right)+\left(\frac{M_{CD}+M_{DC}}{h2}\right) P1 + P2) = o \qquad (3)$$

Example 10:

Find the B.M.D for the shown structure.

Solution:-

$\theta_D = \theta_E = o$

$\theta_C = - \theta_C$

$\theta_B = - \theta_B$

1- **Unknown displacements are:** θB , θC , Δ

2- Relative Stiffness:

AB : BE : BC : CD : ED

$$\frac{1}{5} : \frac{2}{3} : \frac{1}{5} : \frac{1}{3} : \frac{1}{3}$$

$$3 : 10 : 3 : 5 : 5$$

3- Fixed end moment:-

$$MF_{BE} = -\frac{4\times 36}{12} = -12 \text{ t.m}$$

$MF_{EB} = + 12$ t.m

$$MF_{CD} = \frac{1.5\times36}{12} = -4.5 \text{ t.m}$$

$MF_{DC} = +4.5$

4- Equilibrium equations:-

1- $M_{CD} + M_{CB} = o$

2- $M_{BC} + M_{BA} + M_{BE} = o$

3- Shear condition:$(33–16.5)+\frac{M_{CD}+M_{DC}}{6}+\frac{M_{DE}+M_{ED}}{6}$

$$M_{CD} = -4.5 + 5\ (2\theta_C + \theta_D - 3R)$$

$$M_{CB} = 0\ 3\ (2\theta_C + \theta_B)$$

$$M_{BC} = 0 + 3\ (2\theta_B + \theta_C)$$

$$M_{BA} = 0 + 3\ (2\theta_B)$$

$$M_{BE} = -12 + 10\ (2\theta_B - 3R)$$

Hence

$$-4.5 + 10\ \theta_C - 15\ R + 6\ \theta_C + 3\ \theta_B = 0$$

$$16\theta_C + 3\theta_B - 15R - 4.5 = 0 \qquad (1)$$

And

$$16\theta_B + 3\theta_C + 6\theta_B - 12 + \theta_B - 3\theta_R = 0$$

$$3\theta_C + 32\theta_B - 30R - 12 = 0 \qquad (2)$$

and

16.5($\frac{-}{6}$ 30R + 30 θ_B − 60R) =
0 6
15θ_C

$2.5\ \theta_C + 5\theta_C + 17R + 16.5 = 0$ (3)

by solving equation 1,2 & 3 get

$M_{AB} = +6.66$ t.m

$M_{BA} = +13.32$ t.m

$M_{BC} = +19.0$ t.m

$M_{CB} = +18$ t.m M_{BE}

$= -32.32$ t.m $M_{EB} = -$

30.53 t.m $M_{CD} = -$

18 t.m $M_{DC} = -$

18.43 t.m

3-7 **<u>Frame with multiple degree of freedom in translation.</u>**

<u>Example 11</u>:

Write the shown equations and condition eqns for the given frame.

<u>Solution</u>

<u>Unknowns:</u> θB , θC , θD , θE , $\Delta 1$, $\Delta 1$

Condition eqns

$$M_{BE} + M_{BA} + M_{BC} = 0 \quad (1)$$

$$M_{CB} + M_{CD} = 0 \quad (2)$$

$$M_{DC} + M_{DE} = 0 \quad (3)$$

$$M_{EB} + M_{EF} + M_{ED} = 0 \quad (4)$$

Shear eqns :

Equilibrium of the two stories.

At sec (1) – (1) :-

(Level CD)

$$P_2 + X_c + X_E = 0$$

$$P_2 + M_{CB} + M_{BC} + M_{DE} + M_{ED} = 0$$

$h2$ $h2$

At sec. (2) – (2):-

(Level BE) or $\sum x = 0$

$$P_1 + P_2 + x_A + x_F = 0$$

$$P_1 + P_2 + \frac{M_{BA} + M_{AB} + M_{EF} + M_{FE}}{} = 0$$

$h1$ $h1$

Example 12:-

Draw B.M.D for the given structure.

S

olution:-

1- <u>Relative Stiffness:-</u>

2- <u>Equilibrium equations:-</u>

$$M_{AB} + M_{AC} = 0 \qquad (1)$$

$$M_{BA} + M_{BD} = 0 \qquad (2)$$

$$M_{CA} + M_{CD} + M_{CE} = 0 \qquad (3)$$

$$M_{DB} + M_{DF} + M_{DC} = 0 \qquad (4)$$

$\Sigma x = 0$ at Level A-B

$$2 + (6-3) + \frac{M_{AC} + M_{CA}}{6} + \frac{M_{BD} + M_{DB}}{6} = 0 \quad (5)$$

$\Sigma x = 0$ at Level CD

$$11 + \frac{M_{CE} + M_{EC}}{6} + \frac{M_{DF} + M_{FD}}{6} = 0 \quad (6)$$

$M_{AB} = -8 + 1\,(2\theta_A + \theta_B)$ M_{AC}

$= 3 + (2\theta_A + \theta_C - 3R_1)$

$M_{AC} = -3 + (2\theta_C + \theta_A - 3R_1)$

$M_{CA} = 16 + (2\theta_B + \theta_A)$

$M_{BD} = 0 + (2\theta_B + \theta_D – 3R_1)$

$M_{DB} = 0 + (2\theta_D + \theta_B - 3R_1)$

$M_{DF} = 0 + (2\theta_D + 0 - 3R_2)$

$M_{FD} = 0 + (\theta_D - \quad 3R_2)$

$M_{CD} = - 48 + 2 (2\theta_C + \theta_C)$

$M_{CD} = + 48 + 2 (2\theta_D + \theta_C)$

$M_{CE} = - 8 + (2\theta_C - 3R_2)$

$M_{EC} = + (\theta_C - 3R_2)$

3- <u>Fixed end moment:-</u>

$$MF_{AB} = \frac{-9 \times 4 \times 8 \times 4}{12 \times 12} = -8 \text{ t.m}$$

$$MF = -9 \times 8^2 \times 4 = + \quad t.m$$

BA 12 2 16

$$MF_{AC} = \frac{1 \times 6^2}{12} = +3 \text{ t.m}$$

$$MF = \quad 1 \times 6^2 = - \quad t.m$$

CA 12 3

$$MF_{CD} = \frac{4\times 12^{2}}{12} = -48 \quad t.m$$

MF_{DC} = + 48 t.m

4- Unknown displacement:

θA , θB , θC , θD , Δ_1 , Δ_2

by Solving the six equations one can get;

M_{AB} = - 3.84 t.m

M_{BA} = + 18.39 t.m

M_{AC} = 3.84 t.m

M_{CA} = + 7.29 t.m

M_{BD} = - 18.39 t.m

M_{DB} = - 22.97 t.m

M_{CD} = - 11.15 t.m

M_{DC} = - 53.44 t.m

M_{CE} = 3.87 t.m

M_{EC} = - 13.44 t.m

M_{DF} = - 30.47 t.m

M_{FD} = - 26.15 t.m

Example (13):-

Write the shear equations & equilibrium equations for the shown frame.

Solution:

Shear eqns:

$X_{CE} + X_{BA} + P_1 = 0$.................................... (1)

$$\frac{M_{EC} + M_{CE}}{h_1} + \frac{M_{AB} + M_{BA}}{h_1 + h_2} + P_1 = o$$

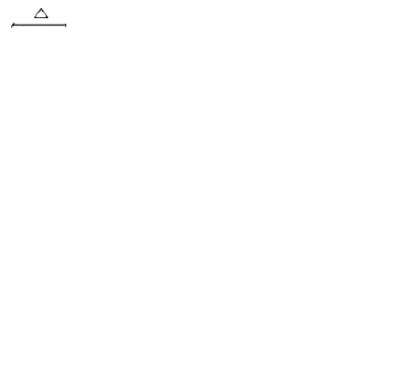
$x_D + x_G + x_E + P_2 = 0$ --- (2)

$$\frac{M_{DE} + M_{ED}}{h_2} + \frac{M_{GF} + M_{FG}}{h_2} - \frac{M_{EC} + M_{CE}}{h_1} + P_2 = 0$$

Or:

$$X_A + X_D + X_G + P_1 + P_2 = 0$$

$$\frac{M_{AB}}{h_1 + h_2} + \frac{M_{DE} + M_{ED}}{h_2} + \frac{M_{GF} + M_{FG}}{h_2} + P_1 + P_2 = 0$$



<u>Example 14:-</u>

a- Write the equations of equilibrium including the shear equations for the frame.

b- Write the slope deflection equations in matrix for members CE & GH.

c- By using the slope - deflection method; sketch elastic curve.

d- Sketch your expected B.M.D

Solution:-

(**Unknowns** = θ_C , θ_D , θ_E , θ_F , θ_G , $\theta_A + \theta_K$, θ_L , Δ_1 ,

Δ_2 , Δ_3 , Δ_4

Relative stiffness: 1 : 1

a- equilibrium equations

$$M_{KL} + M_{KG} = 0 \qquad (1)$$

$$M_{LK} + M_{LH} = 0 \qquad (2)$$

$$M_{GK} + M_{GH} + M_{GE} = 0 \qquad (3)$$

$$M_{HG} + M_{HL} + M_{HF} = 0 \qquad (4)$$

$$M_{EG} + M_{EC} + M_{FF} = 0 \qquad (5)$$

$$M_{FE} + M_{FD} + M_{FH} = 0 \qquad (6)$$

$$M_{CE} + M_{CD} + M_{CA} = 0 \qquad (7)$$

$$M_{DC} + M_{DB} + M_{DF} = 0 \qquad (8)$$

Shear equations:-

a- at Level GH

$$5 + 10 + (X_G - 5) + X_H = 0 \qquad (9)$$

Where:

$$X_G = \frac{M_{GK} + M_{KG}}{5}$$

$$X_H = \frac{M_{HL} + M_{LH}}{5}$$

b- at Level EF

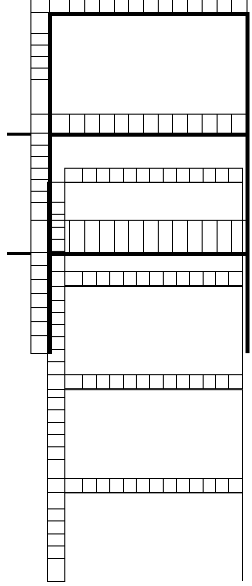
$5 + 10 + 20 + (X_E - 5) + X_F \quad = 0$

$30 + X_E + X_F \quad = 0 \qquad (10)$

Where:

$$X_E = \frac{M_{EG} + M_{GE}}{5}$$

$$X_F = \frac{M_{FH} + M_{HF}}{5}$$



c- at Level CD

$5 + 10 + 10 + 30 + (X_C - 5) + X_D \quad = 0$

$50 + X_C + X_D \quad = 0$ ………(11)

Where:

$$X_C = \frac{M_{CE} + M_{EC}}{5}$$

$$X_D = \frac{M_{DF} + M_{FD}}{5}$$

d- at Sec AB:-

$$5 + 10 + 10 + 10 + 40 + (X_A - 5) + X_B \qquad = 0$$

$$70 + X_A + X_B = 0 \quad \ldots\ldots(12)$$

$$X_A = \frac{M_{AC} + M_{CA}}{5}$$

$$X_B = \frac{M_{CE} + M_{EC}}{5}$$

3-8 Slope deflection eqns in matrix form:

1- Member CE

$$M_{CD} = MF_{CE} + \frac{2EI}{5}\left(2\theta_C + \theta_E - 3\frac{\Delta 2 - \Delta 1}{5}\right)$$

$$M_{EC} = MF_{EC} + \frac{2EI}{5}\left(2\theta_E + \theta_C - 3\frac{\Delta 2 - \Delta 1}{5}\right)$$

Where:

$$MF_{CE} = - \frac{2 \times 5^2}{12} = -4.16 \quad t.m$$

$MF_{EC} = +4.16 \quad t.m$

In Matrix form:

$$\begin{vmatrix} MCE \\ \\ MEC \end{vmatrix} = \begin{vmatrix} -4.16 \\ \\ 4.16 \end{vmatrix} + \frac{2EI}{5} \begin{vmatrix} 2 & 1-3 \\ \\ 1 & 2-3 \end{vmatrix} \begin{vmatrix} \theta_C \\ \theta_E \\ R_2 \end{vmatrix}$$

<u>Where:</u>

$$R_2 = \frac{\Delta_2 - \Delta_1}{5}$$

2- member GH

$$\begin{vmatrix} M_{GH} \\ \\ M_{HG} \end{vmatrix} = \begin{vmatrix} -16.67 \\ \\ +16.67 \end{vmatrix} + \frac{26I}{5} \begin{vmatrix} 2 & 1 \\ \\ 1 & 2 \end{vmatrix} \begin{vmatrix} \theta_G \\ \\ \theta_H \end{vmatrix}$$

d- B.M.D

<u>Example 15:-</u>

By using slope deflection method;

1- Draw B.M.D for the shown frame.

2- Sketch elastic curve.

Solution:

1- Relative stiffness 1: 1

2- unknowns: $\theta_B = -\theta_B$ (From symmetry)

3- Equilibrium eqns

$M_{BA} + M_{BC} + M_{BD} + M_{BB} = o$ (1)

4-Fixed end moments

$$MF_{AB} = \frac{4 \times 6^2}{12} = -12 \text{ t.m}$$

$MF_{BA} =$ $= +12$

$MF_{BC} = MF_{CB} = MF_{BD} = MF_{DE} = 0$

$$MF_{BB} = -\frac{2\times 12^2}{12} - \frac{8\times 12}{8} = -36 \quad t.m$$

4- Slope deflection eqns

$M_{AB} = -12 + (\theta B)$

$M_{BA} = 12 + 2\theta B$

$M_{BC} = 2\theta B$

$M_{BD} = 2\theta B$

$M_{BB} = - 36 + \theta B$

$M_{CB} = \theta B$

$M_{DB} = \theta B$

From eqn (1)

$$(12 + 2\theta_B) + (2\theta_B) + (2\theta_B) + (-36 + \theta_B) = 0$$

$$7\theta_B - 24 = 0$$

$$\theta_B = 3.4286$$

hence

M_{AB} = - 8.57 t.m

M_{BA} = 18.86 t.m

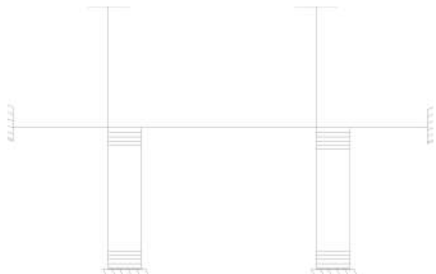
M_{BC} = 6.86 t.m

M_{BB} = - 32.58 t.m

M_{CB} = 3.428 t.m

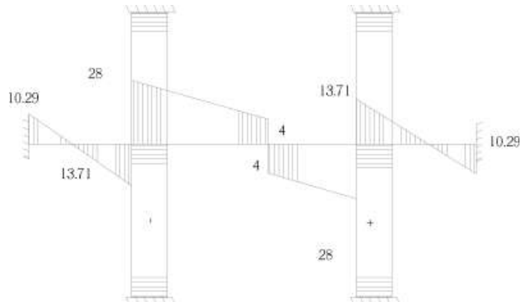
$M_{DB} = 3.428$ t.m

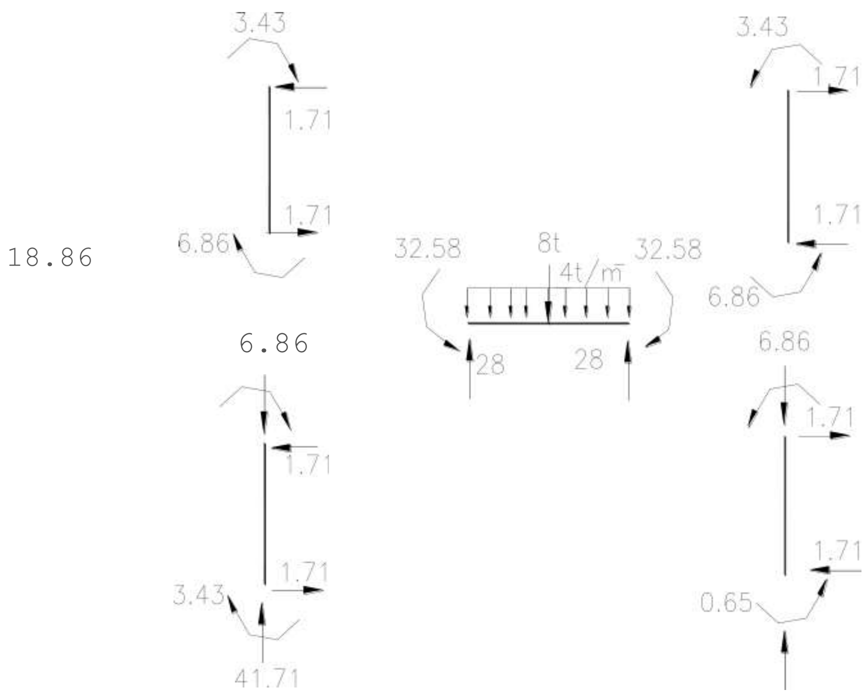
$M_{BD} = 6.86$



41.11 41.71

1.71 1.7 1


28
10.29
13.71
4
4
13.71
10.29
28
+


3.43
1.71
1.71
6.86
18.86
32.58
8t
4t/m
32.58
28
28
3.43
1.71
1.71
6.86
6.86
1.71
1.71
3.43
41.71
6.86
1.71
1.71
0.65

The Free Body Diagram to find the S. F. & N. F.

<u>SHEET (3)</u>

1) Draw S.F.D. and B.M.D. for the statically indeterminate beams shown in figs. From 1 to 10.

2) Draw N.F.D., S.F.D. for the statically indeterminate frames shown in figs. 11 to 17. Using matrix approach 1.