GUDLAVALLERU ENGINEERING COLLEGE

(An Autonomous Institute with Permanent Affiliation to JNTUK, Kakinada)

Seshadri Rao Knowledge Village, Gudlavalleru – 521 356.

Department of Civil Engineering

HANDOUT

on

LINEAR ALGEBRA & DIFFERENTIAL EQUATIONS

Vision

To provide quality education embedded with knowledge, ethics and advanced skills and preparing students globally competitive to enrich the civil engineering research and practice.

Mission

- To aim at imparting integrated knowledge in basic and applied areas of civil engineering to cater the needs of industry, profession and the society at large.
- To develop faculty and infrastructure making the department a centre of excellence providing knowledge base with ethical values and transforming innovative and extension services to the community and nation.
- To make the department a collaborative hub with leading industries and organizations, promote research and development and combat the challenging problems in civil engineering which leads for sustenance of its excellence.

Program Educational Objectives

PEOI : Exhibit their competence in solving civil engineering problems in practice, be

employed in industries and undergo higher studies.

PEOII : Adapt to changing technologies with societal relevance for sustainable development

in the field of their profession.

PEO III: Develop multidisciplinary team work with ethical attitude &social responsibility and

engage in life - long learning to promote research and development in the profession.

LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS

Class & Sem.: I B.Tech – II Semester Year : 2018-19
Branch : CE Credits : 3

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1. **Brief History and Scope of the Subject**

   "MATHEMATICS IS THE MOTHER OF ALL SCIENCES", It is a necessary avenue to scientific knowledge , which opens new vistas of mental activity. A sound knowledge of engineering mathematics is essential for the Modern Engineering student to reach new heights in life. So students need appropriate concepts, which will drive them in attaining goals.

   <u>Scope of mathematics in engineering study :</u>

   Mathematics has become more and more important to engineering Science and it is easy to conjecture that this trend will also continue in the future. In fact solving the problems in modern Engineering and Experimental work has become complicated, time – consuming and expensive. Here mathematics offers aid in planning construction, in evaluating experimental data and in reducing the work and cost of finding solutions.

   The most important objective and purpose in Engineering Mathematics is that the students becomes familiar with Mathematical thinking and recognize the guiding principles and ideas "Behind the science" which are more important than formal manipulations. The student should soon convince himself of the necessity for applying mathematical procedures to engineering problems.

2. **Pre-Requisites**

   Basic Knowledge of Mathematics such as differentiation and Integration at Intermediate Level is necessary.

3. **Course Objectives:**
   - To know different procedures to solve the system of linear equations.
   - To find the Eigenvalues and Eigenvectors.
   - To find the solutions of 1^{st} and 2^{nd} order Differential equations.

4. **Course Outcomes:**

   Students will be able to

- to gain the knowledge of Laplace and their inverse transforms
- to study the Fourier Transform concepts.
- to know vector integral theorems such as Green's, Gauss & Stoke's theorems.

5. **Program Outcomes:**

   Graduates of the Civil Engineering Program will have

a) An ability to apply knowledge of mathematics, science and engineering principles to civil engineering problems.
b) An ability to analyze design and conduct experiments and interpret the resulting data.
c) An ability to design a system, component or process to meet desired goals in civil engineering applications.
d) An ability to function on multi disciplinary teams.
e) An ability to identify, formulate and solve challenging engineering problems.
f) An understanding of professional and ethical responsibility.
g) An ability to communicate effectively through verbal, written and drawing presentations.
h) An ability to understand the impact of engineering solutions in a global, economical and social context with a commitment on environmental and safety issues.
i) An ability to recognize the need of engaging in lifelong learning and acquiring further knowledge in specialized areas.
j) Ability to excel in competitive examinations, advanced studies and become a successful engineer in construction industry.
k) An ability to use the techniques, skills and modern engineering tools and software for engineering design and practices.
l) The understanding of basic finance & management techniques and construction practices including work procurement and legal issues.

6. Mapping of Course Outcomes with Program Outcomes:

	a	b	c	d	e	f	g	h	i	j	k	l
CO1	**H**	**M**										
CO2	**M**	**L**										
CO3	**M**	**L**										
CO4	**H**	**M**										

7. Prescribed Text Books

1. B.S.Grewal, Higher Engineering Mathematics : 42^{nd} edition, Khanna Publishers,2012 , New Delhi.
2. B.V Ramana, Higher Engineering Mathematics, Tata-Mc Graw Hill Company Ltd.

8. Reference Text Books

1. U.M.Swamy, A Text Book of Engineering Mathematics – I & II : 2^{nd} Edition, Excel Books, 2011, New Delhi.
2. Erwin Kreyszig, Advanced Engineering Mathematics : 8th edition, Maitrey Printech Pvt. Ltd, 2009, Noida.

9. URLs and Other E-Learning Resources

Sonet CDs & IIT CDs on some of the topics are available in the digital library.

10. Digital Learning Materials:

- **http://nptel.ac.in/courses/106106094**
- **http://nptel.ac.in/courses/106106094/40**
- **http://nptel.ac.in/courses/106106094/30**
- **http://nptel.ac.in/courses/106106094/32**
- **http://textofvideo. nptl.iitm.ac.in/106106094/lecl.pdf**

11. Lecture Schedule / Lesson Plan

S.No	TOPIC	No of. Periods	No of. Tutorials
	UNIT-I		

1	Laplace transforms of standard functions	1	1
2	Shifting Theorems	1	
3	change of scale	1	
4	Transforms of derivatives	1	
5	Transforms of integrals	1	1
6	Unit step function –Dirac's delta function	1	
7	Evaluation of Improper Integrals	2	
8	Review and conclusion	1	
	UNIT-II		
9	Inverse Laplace transforms	1	1
10	Inverse Laplace transforms by partial fractions	2	
11	Convolution theorem (with out proof).	1	1
12	Inverse Laplace transforms by Convolution theorem	2	
13	Solutions of ordinary differential equations using Laplace transforms	3	
	UNIT-III		
14	Fourier integral theorem & Problems	2	1
15	Properties of Fourier transform (without proofs)	3	
16	Fourier transform, sine and cosine transforms & Problems	1	
17	Inverse Fourier transforms	2	
18	Review and conclusion	1	1
	UNIT-IV		
19	Double integrals	2	1
20	Finding area using double integral	3	
21	Triple integrals	2	1
22	Finding Volume using Triple integrals	3	
	UNIT-V		
23	Gradient and its properties	1	1
24	Directional derivatives and problems	2	
25	Divergence and Curl introduction	1	
26	Scalar Potential and Problems	1	1
27	Del applied twice	1	
28	Laplacian Operator and Problems	2	
	UNIT-VI		
29	Apply greens theorem to evaluate line integrals	3	1
30	Apply stokes theorems to evaluate surface and volume integrals	2	
31	Apply divergence theorems to evaluate surface and volume integrals	2	1

12. Seminar Topics

- **Evaluation of Improper Integrals**
- **Convolution theorem**
- **Fourier integral theorem**

INTEGRAL TRANSFORMS AND MULTIPLE INTEGRALS

UNIT-I: LAPLACE TRANSFORMS

Objectives:

- **To know the properties of Laplace transforms**
- **To know the Transform of one variable function to another variable function.**
- **To find the Laplace Transform of standard functions**

Syllabus: Laplace transform of standard functions- Properties: Shifting Theorems, change of scale, derivatives, integrals, multiplication and division – Unit step function – Dirac Delta function, Evaluation of improper integrals.

Course Outcomes:

The students is able to

- **Calculate the Laplace transform of standard functions both from the definition and by using formulas**
- **Select and use the appropriate shift theorems in finding Laplace transforms.**
- **Evaluation of Improper integrals.**

Introduction:

The Laplace Transformation

Pierre-Simon Laplace (1749-1827)

Laplace was a French **mathematician**, **astronomer**, and **physicist** who applied the Newtonian theory of gravitation to the solar system (an important problem of his day). He played a leading role in the development of the **metric system**.

The **Laplace Transform** is widely used in **engineering applications** (mechanical and electronic), especially where the driving force is discontinuous. It is also used in process control.

Laplace Transform (**LT**) is a powerful technique to replace the operations of calculus by operations of algebra.

***Definition*:** Let f be a function defined for t ≥ 0.We define Laplace transform of f ,denoted by F(s) or L{f(t)) or $\overline{f}$ (s) as F(s) = L{f(t)}= $\int_0^\infty e^{-st} f(t)dt$ for those s for which the integral exists is called the Laplace Transform or one sided Laplace Transform.

Sufficient conditions for the existence of L.T:

1) f is piecewise continuous on the interval 0 ≤ t ≤ A for any A > 0.

2) f is of exponential order i.e., If $f(t)$ is defined for all $t > 0$ and there exists constants α and M such that $|f(t)| \le Me^{\alpha t}$ for all t.

- *Note (1):* One sided LTs are unilateral whereas two sided LTs are bilateral Laplace Transforms.
- *Note (2):* A two sided LT obtained by setting the other limit of integral as $-\infty$.

Laplace transforms of some elementary functions:

Let $f(t) = 1$ then $L\{f(t)\} = L(1) = \frac{1}{s}, \ \ s > 0$

1. Let $f(t) = e^{at}$ then $L\{f(t)\} = L(e^{at}) = \frac{1}{s-a}, \ \ s > a$
2. Let $f(t) = e^{-at}$ then $L\{f(t)\} = L(e^{-at}) = \frac{1}{s+a}, \ \ s > -a$.
3. Let $f(t) = t^n$ then $L\{f(t)\} = L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$.
4. Let $f(t) = \sin at$ then $L\{f(t)\} = L(\sin at) = \frac{a}{s^2 + a^2}, s > 0.$
5. Let $f(t) = \cos at$ then $L\{f(t)\} = L(\sin at) = \frac{s}{s^2 + a^2}, s > 0.$
6. Let $f(t) = \sinh at$ then $L\{f(t)\} = L(\sinh at) = \frac{a}{s^2 - a^2}, s > |a|.$

7. Let $f(t) = \cosh at$ then $L\{f(t)\} = L(\sin at) = \frac{s}{s^2 - a^2}$, $s > |a|$.

Properties of Laplace transform:

1. Laplace transform operator L is linear. Laplace transform of a linear combination (sum) of functions is the linear combination (sum) of Laplace transforms of the functions.
2. Change of scale property: When the argument t of f is multiplied by a constant k, s is replaced by s/k in $\bar{f}(s)$ or $F(s)$ and multiplied by $1/k$.
3. First shift theorem proves that multiplication of $f(t)$ by e^{at} amounts to replacement of s by $s-a$ in $\bar{f}(s)$.
4. Laplace transform of a derivative f' amounts to multiplication of $\bar{f}(s)$ by s (approximately but for the constant $-f(0)$).
5. Laplace transform of integral of f amounts to division of $\bar{f}(s)$ by s.
6. Laplace transform of multiplication of $f(t)$ by t^n amounts to differentiation of $\bar{f}(s)$ for n times w.r.t. s (with $(-1)^n$ as sign).
7. Division of $f(t)$ by t amounts to integration of $\bar{f}(s)$ between the limits s to ∞.
8. Second shift theorem proves that the L.T. of shifted function $f(t-a)u(t-a)$ is obtained by multiplying $\bar{f}(s)$ by e^{-at}.

Problems:

1) If f(t) = t^3+4t^2+5, then L[f(t)] = $\frac{\Gamma(4)}{s^4} + 4\frac{\Gamma(3)}{s^3} + 5\frac{\Gamma(2)}{s^2} = \frac{6}{s^4} + \frac{8}{s^3} + \frac{5}{s^2}$

2) Find Laplace transform of $\sin t \cos 2t$.

Solution: Let $f(t) = \sin t \cos 2t$

$$= \frac{1}{2}(\sin 3t - \sin t)$$

Apply LT on both sides, we have

$L(\sin t \cos 2t) = L\left[\frac{1}{2}(\sin 3t - \sin t)\right] = \frac{1}{2}L(\sin 3t) - \frac{1}{2}L(\sin t)$ (Using linearity property of LT)

$$= \frac{1}{2}\left(\frac{3}{s^2+9}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right).$$

3) Find the LT of $e^{-4t}\sin 3t$.

Solution: Let $f(t) = \sin 3t$

By the definition of LT, $L\{\sin 3t\} = \frac{3}{s^2+a^2}$

Hence by first shifting theorem, $L\{e^{-4t}\sin 3t\} = \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+25}$.

Laplace transforms of derivatives:

Statement: Let $f(t)$ be a real continuous function which is of exponential order and $f'(t)$ is sectionally continuous and is of exponential order. Then $L\{f'(t)\} = s\bar{f}(s) - f(0)$ Where $\bar{f}(s) = L\{f(t)\}$.

In general,

$L\{f^{(n)}(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0).$

Laplace transforms of integrals:

Statement: Suppose $f(t)$ is a real function and $g(t) = \int_0^t f(u)du$ is a real function such that both $f(t), g(t)$ satisfy the conditions of existence of Laplace transform then

$L\{g(t)\} = L\left[\int_0^t f(u)du\right] = \frac{\bar{f}(s)}{s}$ Where $\bar{f}(s) = L\{f(t)\}$.

Laplace transform of the function $f(t)$ multiplied by t^n:

Statement: If $f(t)$ is sectionally continuous and is of exponential order and if $L\{f(t)\} = \bar{f}(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}$ where $n = 1,2,....$

Laplace transform of the function $f(t)$ divided by t^n:

If $L\{f(t)\} = \bar{f}(s)$ then $L\left(\frac{f(t)}{t}\right) = \int_0^\infty \bar{f}(s)ds$ provided $f(t)$ satisfy the condition of existence of LT and the right hand side integral exists.

4) *Problem:* Find the Laplace transform of $f(t) = t\cosh at$, using LT of derivatives.

Solution: We are given $f(t) = t\cosh at$.

It is known that $f^{'}(t) = a\cosh at + at\sinh at$ and

$$f^{''}(t) = 2a\sinh at + a^2 t\cosh at$$

By applying LT on both sides, $L\{f^{''}(t)\} = 2aL\{\sinh at\} + a^2 L\{t\cosh at\}$

By the LT of derivatives, $s^2 L\{f(t)\} - sf(0) - f^{'}(0) = 2a\dfrac{a}{s^2 - a^2} + a^2 L\{t\cosh at\}$

Since $f(0) = 0$ and $f^{'}(0) = 1$, on simplification, we have

$$L\{t\cosh at\} = \frac{2a^2}{(s^2 - a^2)^2}.$$

5) ***Problem:*** Find $L\left(\int_0^t ue^{-u}\sin 4u\,du\right)$.

Solution: Let $f(t) = \sin 4u$

By LT, $L\{\sin 4u\} = \dfrac{4}{s^2 + 4^2} = \dfrac{4}{s^2 + 16}$

By first shifting theorem, $L\{e^{-u}\sin 4u\} = \dfrac{4}{(s+1)^2 + 16} = \dfrac{4}{s^2 + 2s + 17}$

Then by LT of $t^n f(t)$, $L\{ue^{-u}\sin 4u\} = -\dfrac{d}{ds}\left(\dfrac{4}{s^2 + 2s + 17}\right) = \dfrac{4}{(s^2 + 2s + 17)} = \bar{f}(s)$.

Therefore, the LT of integrals, we have

$$L\left(\int_0^t ue^{-u}\sin 4u\,du\right) = \frac{\bar{f}(s)}{s} = \frac{4}{s(s^2 + 2s + 17)}.$$

6) ***Problem:*** Find $L\left(\dfrac{\sin at\cos bt}{t}\right)$.

Solution: Let $f(t) = \sin at\cos bt$

$$= \frac{1}{2}\left[\sin(a+b)t + \sin(a-b)t\right]$$

By applying LT on both sides,

$$L\{\sin at\cos bt\} = \frac{1}{2}\left[L\{\sin(a+b)t\} + L\{\sin(a-b)t\}\right]$$

$$= \frac{1}{2}.\frac{(a+b)}{s^2 + (a+b)^2} + \frac{1}{2}.\frac{(a-b)}{s^2 + (a-b)^2} = \bar{f}(s)$$

Now, by the LT of $\frac{f(t)}{t}$, $L\left\{\frac{\sin at\cos bt}{t}\right\}=\frac{1}{2}\int_s^\infty \frac{(a+b)}{k^2+(a+b)^2}ds+\frac{1}{2}\int_s^\infty \frac{(a-b)}{k^2+(a-b)^2}ds$

$$=\frac{1}{2}\left[\tan^{-1}\left(\frac{k}{a+b}\right)\right]_s^\infty+\frac{1}{2}\left[\tan^{-1}\left(\frac{k}{a-b}\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{a+b}\right)\right]+\frac{1}{2}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{a-b}\right)\right]$$

$$=\frac{1}{2}\cot^{-1}\left(\frac{s}{a+b}\right)+\frac{1}{2}\cot^{-1}\left(\frac{s}{a-b}\right).$$

Unit Step function:

Definition: Unit step function is defined as $U(t-a)=0,\ \ t<a$

$=1,\ t>a$ i.e. this function jumps by1 at $t=a$.

This function is also known as Heaviside unit function.

Laplace transform of Unit step function $U(t-a)$ is given by

$$L\{U(t-a)\}=\int_0^\infty e^{-st}U(t-a)dt=\int_0^a e^{-st}.0dt+\int_a^\infty e^{-st}.1dt=\int_a^\infty e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_a^\infty=\frac{e^{-as}}{s}.$$

Unit impulse function:

Definition: The unit impulse function denoted by $\delta(t-a)$ and is defined by

$$\delta(t-a)=\infty,\ \ t=a$$
$$=0,\ \ t\neq a$$

So that $\int_0^\infty \delta(t-a)dt=1 \quad (a\geq 0)$.

If a moving object collide with another object then for a short period of time large force is acting on the other body. To explain such mechanism we make use of unit impulse function, which is also called Dirac Delta function.

Evaluation of improper integrals by Laplace transforms:

Problem: Evaluate the integral, $\int_0^\infty \frac{\cos at-\cos bt}{t}dt$.

Solution: Let $I = \int_0^\infty \frac{\cos at - \cos bt}{t} dt$.

$$= \int_0^\infty \frac{\cos at}{t} dt - \int_0^\infty \frac{\cos bt}{t} dt$$

Clearly the given integral is in the form $\int_0^\infty e^{-st} \frac{f(t)}{t} dt$ with $f_1(t) = \cos at$ and

$f_1(t) = \cos bt$

We observe that $\int_0^\infty e^{-st} \frac{\cos at}{t} dt = \int_s^\infty L(\cos at) ds = \int_s^\infty \frac{s}{s^2 + a^2} ds$ and

$$\int_0^\infty e^{-st} \frac{\cos bt}{t} dt = \int_s^\infty L(\cos bt) ds = \int_s^\infty \frac{s}{s^2 + b^2} ds$$

$$\therefore \int_0^\infty e^{-st} \left(\frac{\cos at - \cos bt}{t} \right) dt = \int_0^\infty \frac{s}{s^2 + a^2} ds - \int_0^\infty \frac{s}{s^2 + b^2} ds = \int_0^\infty \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right] ds$$

It is clear that the above integral reduces to I when $s = 0$.

Therefore,

$$I = \int_0^\infty \frac{\cos at - \cos bt}{t} dt = \int_0^\infty \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right] ds = \left[\frac{1}{2} \log(s^2 + a^2) - \frac{1}{2} \log(s^2 + b^2) \right]_0^\infty$$

$$= \frac{1}{2} \left[\log \left(\frac{s^2 + a^2}{s^2 + b^2} \right) \right]_0^\infty = \frac{1}{2} \left[\log 1 - \log \left(\frac{a^2}{b^2} \right) \right] = \frac{1}{2} \log \left(\frac{a^2}{b^2} \right).$$

Assignment/Tutorial Questions

SECTION-A

1. The Laplace transform of $f(t) = \sin^2 2t$ is ________________.
2. If $f(t) = e^{3t}(\sin 2t + \cos 3t)$ then $L\{f(t)\} =$ ________________.
3. If $f(t) = \frac{e^{2t} - e^{3t}}{t}$ then $L\{f(t)\} =$ ________________.
4. If $f(t) = t \sin t$ then $L\{f(t)\} =$ ________________.
5. The value of $\int_0^\infty e^{-3t} t\,dt$ is ________________.
6. $L\{e^{at} t^n\} =$ ________________.
7. The Laplace transform of $\frac{(1 - e^t)}{t}$ is ________________.
8. If $L\{f(t)\} = \bar{f}(s) = \frac{s}{s^2 + 1}$, $f(0) = 0$ then $L\{f'(t)\} =$ ________________.
9. Find the Laplace transform of $t^{5/2}$

   (a) $\frac{15}{8}\frac{\sqrt{\pi}}{s^{\frac{5}{2}}}$ (b) $\frac{15}{8}\frac{\sqrt{\pi}}{s^{\frac{7}{2}}}$ (c) $\frac{9}{4}\frac{\sqrt{\pi}}{s^{\frac{7}{2}}}$ (d) $\frac{15}{4}\frac{\sqrt{\pi}}{s^{\frac{7}{2}}}$

10. Laplace transform of f(t) is given by

    a) $F(s) = \int_{-\infty}^{\infty} f(t) e^{-st}\,dt$

    b) $F(t) = \int_{-\infty}^{\infty} f(t) e^{-t}\,dt$

    a) $f(s) = \int_{-\infty}^{\infty} f(t) e^{-st}\,dt$

    b) $f(t) = \int_{-\infty}^{\infty} f(t) e^{-t}\,dt$

11. Laplace transform of sin(at)u(t) is

    a) s/a^2+s^2
    b) a/a^2+s^2
    c) s^2/a^2+s^2
    d) a^2/a^2+s^2

12. Find the laplace transform of $y(t)=e^{|t-1|}$ u(t).

    a) $\frac{2s}{1-s^2} e^{s}$

    b) $\frac{2s}{1+s^2} e^{-s}$

    c) $\frac{2s}{1+s^2} e^{s}$

    d) $\frac{2s}{1-s^2} e^{-s}$

13. Find the Laplace transform of e^t Sin(t).

a) $\frac{a}{a^2+(s+1)^2}$

b) $\frac{a}{a^2+(s-1)^2}$

c) $\frac{s+1}{a^2+(s+1)^2}$

d) $\frac{s+1}{a^2+(s+1)^2}$

SECTION-B

1. Find L[tcosat] by multiplication t property.
2. Find L[cos(at+b)]
3. Find $L[sin^2(2t)]$
4. Find $L[sin2tcos3t]$
5. Find the Laplace transform of $f(t) = \begin{cases} e^t, 0 < t < 1 \\ 0 \quad ,t > 1 \end{cases}$
6. Find the Laplace transform of $\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3$
7. Define Unit-step function and also write its Laplace transform.
8. Define Dirac Delta function.
9. Evaluate $L[t^2e^{-t}cos^2t]$
10. Evaluate $L[\frac{cosat-cosbt}{t}]$
11. Evaluate $L\left[\int_0^t \frac{e^t sint}{t} dt\right]$
12. Evaluate L[tsint] and hence find $L[\int_0^t \int_0^t tsintdt\, dt]$
13. Derive the Laplace transform of Unit Step function and hence find $L[e^{t-3}u(t-3)]]$
14. Evaluate $\int_0^\infty e^{-t} \frac{sin^2t}{t} dt$
15. Evaluate $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t} dt$, using Laplace transform.

GATE PREVIOUS QUESTIONS

1. The Laplace Transform of cos(ωt) is $\frac{s}{s^2+\omega^2}$ then L(e^{-2t} cos4t) is **(GATE-2010)**

(a) $\frac{s-2}{(s-2)^2+16}$ (b) $\frac{s+2}{(s-2)^2+16}$ (c) $\frac{s-2}{(s+2)^2+16}$ (d) $\frac{s+2}{(s+2)^2+16}$

2. The L.T of f(t) = $\frac{1}{s^2(s+1)}$ then f(t) is **(GATE-2010)**

(a) t-1 + e^{-t} (b) t + 1 + e^{-t} (c) -1 + e^{-t} (d) 2t + e^t

3. If L.T of sin wt is $\frac{s}{s^2+w^2}$ then L.T of e^{-2t} .sint is **(GATE-2014)**

(a) $\frac{s-2}{(s-2)^2+16}$ (b) $\frac{s+2}{(s-2)^2+16}$ (c) $\frac{s-2}{(s+2)^2+16}$ (d) $\frac{s+2}{(s+2)^2+16}$

4. If F(s) is the L.T of f(t) then. L.T of $\int_0^t f(\tau)d\tau$ is **(GATE-2007)**

(a) $\frac{1}{s}F(s)$ (b) $\frac{1}{s}F(s)$-f(0) (c)sF(s)-f(0) (d) $\int F(s)ds.$

5. L.T of functions t.u(t) and u(t).sint are respectively. **(GATE-1987)**

(a) $\frac{1}{s^2},\frac{s}{s^2+1}$ (b) $\frac{1}{s},\frac{1}{s^2+1}$ (c) $\frac{1}{s^2},\frac{1}{s^2+1}$ (d) $s,\frac{s}{s^2+1}$

6. The L.T of i(t) is given by I(s) = $\frac{2}{s(1+s)}$ as $t\rightarrow\infty$ the value of i(t) tends to

(a) 0 (b) 1 (c) 2 (d) ∞

7. The unilateral Laplace transform of f (t)=$\frac{1}{s^2+s+1}$ is **(GATE-2012)**

(a) $\frac{-s}{(s^2+s+1)^2}$ (b) $\frac{s}{(s^2+s+1)^2}$ (c) $\frac{-(2s+1)}{(s^2+s+1)^2}$ (d) $\frac{2s+1}{(s^2+s+1)^2}$

INTEGRAL TRANSFORMS AND MULTIPLE INTEGRALS

UNIT-II: INVERSE LAPLACE TRANSFORMS

Objectives:

- To understand the properties of Inverse Laplace transforms
- To solve Integral equations by using convolution theorem.
- To convert differential equations into algebraic equations using Laplace Transforms and inverse Laplace transforms.

Syllabus:

Inverse Laplace Transforms – by partial fractions - Convolution theorem (without proof).

Application: Solution of ordinary differential equations.

Subject Outcomes/Unit Outcomes:

After learning this unit, students will be able to:

- Find inverse Laplace Transforms of the transformation $\bar{f}(s)$ to obtain $f(t)$.
- Apply convolution theorem to find the inverse Laplace
- Use the method of Laplace transforms to solve systems of linear ordinary differential equations.

Definition: Suppose $f(t)$ is a piecewise continuous function and is of exponential order. Let $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \bar{f}(s)$. The inverse Laplace Transform (ILT) of $\bar{f}(s)$ is defined as $L^{-1}\{\bar{f}(s)\} = f(t),$ where L^{-1}inverse operator of is L and vice-versa.

Inverse Laplace transforms of some elementary functions:

(1). $L^{-1}\left\{\frac{1}{s}\right\} = 1$ (2). $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ (3). $L^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\} = t^n$ (4). $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

(5). $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ (6). $L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$ (7). $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$, etc.

Properties of Inverse Laplace transform:

Linear property:

If $L^{-1}\{\bar{f}(s)\} = f(t), L^{-1}\{\bar{g}(s)\} = g(t)$, then $L^{-1}\{a\,\bar{f}(s) + b\,\bar{g}(s)\} = a\,f(t) + b\,g(t)$

Shifting Property:
If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t),\ s > a$.
Change of scale property:
If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{\bar{f}(as\} = \frac{1}{a}\bar{f}\left(\frac{t}{a}\right)$ *and* $L^{-1}\left\{\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)\right\} = f(at)$

Problem: let $\bar{f}(s) = \frac{4s+4}{4s^2-9}$. Then by linearity property of inverse Laplace transforms (ILT),

$$L^{-1}\left\{\frac{4s+4}{4s^2-9}\right\} = L^{-1}\left\{\frac{4s}{4s^2-9}\right\} + L^{-1}\left\{\frac{4}{4s^2-9}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2-(3/2)^2}\right\} + L^{-1}\left\{\frac{1}{s^2-(3/2)^2}\right\} = \cosh\frac{3}{2}t + \frac{2}{3}\sinh\frac{3}{2}t$$

Problem: Find the ILT of $\frac{4}{(s+1)(s+2)}$.

Solution: Let $\bar{f}(s) = \frac{4}{(s+1)(s+2)}$

By applying partial fractions, we can rewrite $\bar{f}(s)$ as

$$\bar{f}(s) = \frac{4}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{As+2A+Bs+B}{(s+1)(s+2)}$$

Comparing like terms in the numerator, we obtain $A = 4$ and $B = -4$.

Therefore, $\bar{f}(s) = \frac{4}{(s+1)(s+2)} = \frac{4}{(s+1)} - \frac{4}{(s+2)}$

By applying linearity property, we have

$$L^{-1}\{\bar{f}(s)\} = 4L^{-1}\left\{\frac{1}{s+1}\right\} - 4L^{-1}\left\{\frac{1}{s+2}\right\} = 4e^{-t} - 4e^{-2t}.$$

Problem: Find the ILT of $\frac{s+1}{s^2+s+1}$.

Solution: Consider $\bar{f}(s) = \frac{s+1}{s^2+s+1}$

$$= \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

By the linearity property of ILT, we have

$$L^{-1}\left(\frac{s+1}{s^2+s+1}\right)=L^{-1}\left(\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)+L^{-1}\left(\frac{1/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$=e^{-t/2}\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t=e^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right].$$

Inverse Laplace Transforms of Derivatives:

Statement: If $L^{-1}\{\bar{f}(s)\}=f(t)$ then $L^{-1}\left(\frac{d^n(\bar{f}(s))}{ds^n}\right)=(-1)^n t^n f(t).$

Inverse Laplace Transforms of Integrals:

Statement: If $L^{-1}\{\bar{f}(s)\}=f(t)$ then $L^{-1}\left(\int_s^\infty \bar{f}(s)ds\right)=\frac{f(t)}{t}.$

Inverse Laplace Transform of type $s\bar{f}(s)$: (Multiplication by s)
Statement: If $L^{-1}\{\bar{f}(s)\}=f(t)$ and $f(0)=0$ then $L^{-1}\left(s\bar{f}(s)\right)=f'(t)$

Inverse Laplace Transform of type $\frac{\bar{f}(s)}{s}$: (Division by s)

Statement: If $L^{-1}\{\bar{f}(s)\}=f(t)$ then $L^{-1}\left(\frac{\bar{f}(s)}{s}\right)=\int_0^t f(t)dt$

Similarly, $L^{-1}\left(\frac{\bar{f}(s)}{s^2}\right)=\int_0^t\int_0^t f(t)dt$ and hence in general,

$L^{-1}\left(\frac{\bar{f}(s)}{s^n}\right)=\int_0^t\int_0^t\ldots\int_0^t f(t)dt\,dt\ldots dt$ (n-folded integral).

Problem: Evaluate $L^{-1}\left\{\frac{s}{\left(s^2+2^2\right)^2}\right\}$ using derivative property of ILT.

Solution: We know that $L^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$, then by derivative property of ILT,

we have $L^{-1}\left[\frac{-2s}{\left(s^2+a^2\right)^2}\right]=-\frac{t}{a}\sin at$, $\therefore L^{-1}\left\{\frac{s}{(s^2+2^2)^2}\right\}=\frac{t}{4}\sin 2t$.

Convolution Theorem:-
This is used to find inverse Laplace transforms of product of transforms.
Definition: The convolution of two functions $f(t)$ and $g(t)$ is defined as:

$f(t)*g(t)=\int_0^t f(r)g(t-r)dr$, provided the integral exists.

Note: the operation of convolution between two functions yields another function.

Convolution Theorem:-

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$ then $L^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t) * g(t)$.

Example: Using convolution theorem find the inverse Laplace transform of $\dfrac{s^2}{(s^2+4)(s^2+9)}$.

Solution: We are given $f(t) = \dfrac{s^2}{(s^2+4)(s^2+9)}$

The given function $f(t)$ can be rewritten as,

$$f(t) = \frac{s^2}{(s^2+4)(s^2+9)} = \frac{s}{(s^2+4)} \cdot \frac{s}{(s^2+9)}$$

By applying inverse Laplace transform, we have,

$$L^{-1}\{f(t)\} = L^{-1}\left\{\frac{s^2}{(s^2+4)} \cdot \frac{s^2}{(s^2+9)}\right\}$$

Hence by convolution theorem,

$$L^{-1}\left\{\frac{s^2}{(s^2+4)} \cdot \frac{s^2}{(s^2+9)}\right\} = (\cos 2t) * (\cos 3t)$$ since, $L^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos 2t$ and $L^{-1}\left(\dfrac{s}{s^2+9}\right) = \cos 3t$

$$= \int_0^t [\cos 2u \cos 3(t-u)]du = \int_0^t \frac{1}{2}[\cos(3t-u) + \cos(5u-3t)]du$$

$$= \frac{1}{2}\left[\frac{\sin(3t-u)}{(-1)}\right]_0^t + \frac{1}{2}\left[\frac{\sin(5u-3t)}{5}\right]_0^t = \frac{-1}{2}[\sin 2t - \sin 3t] + \frac{1}{10}[\sin 2t + \sin 3t]$$

$$= \sin 2t\left(-\frac{1}{2} + \frac{1}{10}\right) + \sin 3t\left(\frac{1}{2} + \frac{1}{10}\right) = \frac{1}{5}(3\sin 3t - 2\sin 2t).$$

Solution of Ordinary differential equation (An application):

Problem: Solve the differential equation $\dfrac{d^3y}{dt^3} + 2\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 2y = 0;$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$.

Solution: We are given the linear non-homogeneous differential equation with constant coefficients:

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$ where $y = y(t)$ or $f(t)$

Applying Laplace transform on both sides,

$$L\left(\frac{d^3y}{dt^3}\right) + 2L\left(\frac{d^2y}{dt^2}\right) - L\left(\frac{dy}{dt}\right) - 2L(y) = L(0)$$

$$\Rightarrow [s^3\bar{f}(s) - s^2 f(0) - sy'(0) - y''(0)] + 2[s^2\bar{f}(s) - sy(0) - y'(0)] - [s\bar{f}(s) - y(0)] - 2\bar{f}(s) = 0$$

$$\Rightarrow \bar{f}(s)[s^3 + 2s^2 - s - 2] - y(0)[s^2 + 2s - 1] - y'(0)(s+2) - y''(0) = 0$$

Substituting $y(0)=y^{'}(0)=0$ and $y^{''}(0)=6$, we get,

$$\bar{f}(s)(s^3+2s^2-s-2)-6=0$$

$$\Rightarrow \bar{f}(s)=\frac{6}{(s^3+2s^2-s-2)}$$

Now by applying inverse Laplace transform on both sides,

$$L^{-1}(\bar{f}(s))=L^{-1}\left(\frac{6}{s^3+2s^2-s-2}\right)=L^{-1}\left(\frac{6}{s^2(s+2)-(s+2)}\right)$$

$$f(t)=L^{-1}\left(\frac{6}{(s+2)(s+1)(s-1)}\right)$$

Consider $\bar{f}(s)=\frac{6}{(s-1)(s+1)(s+2)}=\frac{A}{(s-1)}+\frac{B}{(s+1)}+\frac{C}{(s+2)}$

On simplification we obtain $A=1,\quad B=-3,\quad C=2$

$$\therefore\quad L^{-1}(\bar{f}(s))=f(t)=L^{-1}\left(\frac{1}{s-1}\right)-L^{-1}\left(\frac{3}{s+1}\right)+L^{-1}\left(\frac{2}{s+2}\right)$$

$$=e^t-3e^{-t}+2e^{-2t}$$

Hence, the solution of the given differential equation is $y(t)=e^t-3e^{-t}+2e^{-2t}$.

Problem: Solve the differential equation $t\frac{d^2y}{dt^2}+(1-2t)\frac{dy}{dt}-2y=0$ where $y(0)=1,\ y^{'}(0)=2$.

Solution: We are given the linear differential equation with variable coefficients:

$$t\frac{d^2y}{dt^2}+(1-2t)\frac{dy}{dt}-2y=0$$

Applying Laplace transform on both sides,

$$L\left(t\frac{d^2y}{dt^2}\right)+L\left((1-2t)\frac{dy}{dt}\right)-2L(y)=0$$

$$\Rightarrow -\frac{d}{ds}\left(s^2\bar{f}(s)-sf(0)-f^{'}(0)\right)+\left(s\bar{f}(s)-f(0)\right)+2\frac{d}{ds}\left(s\bar{f}(s)-f(0)\right)-2\bar{f}(s)=0$$

$$\Rightarrow \bar{f}^{'}(s)(2s-s^2)-s\bar{f}(s)=0$$

$$\Rightarrow \frac{\bar{f}^{'}(s)}{\bar{f}(s)}=-\frac{1}{s-2}$$

Integrating on both sides, we have,

$$\log\bar{f}(s)=-\log(s-2)+\log c$$

$$\Rightarrow \bar{f}(s)=\frac{c}{s-2}$$

By applying inverse Laplace transform on both sides,

$$L^{-1}(\bar{f}(s))=L^{-1}\left(\frac{c}{s-2}\right)$$

$$\Rightarrow f(t)=ce^{2t}$$

By using the initial condition, we have $c=1$.

Therefore, the particular solution of the differential equation is $f(t) = e^{2t}$.

Assignment/Tutorial Questions
SECTION-A

1. $L^{-1}\left(\frac{1}{s^2+a^2}\right) =$

   (a) $\sin at$ (b) $\cos at$ (c) $\frac{1}{a}\sin at$ (d) $\frac{1}{a}\cos at$

2. $L^{-1}\left(\frac{1}{3s-6}\right) =$

   (a) e^{6t} (b) $\frac{1}{3}e^{2t}$ (c) e^{2t} (d) does not exist

3. $L^{-1}\left(\frac{1}{(s+a)(s+b)}\right) =$

   (a) $\frac{e^{at}-e^{bt}}{b-a}$ (b) $\frac{e^{-at}+e^{-bt}}{b-a}$ (c) $\frac{e^{-at}-e^{-bt}}{b-a}$ (d) $\frac{e^{at}+e^{bt}}{b-a}$

4. $L^{-1}\left(\frac{s+2}{(s-2)^2}\right) =$

   (a) $e^{2t}(1+2t)$ (b) $te^{2t}(1+2t)$ (c) $(1+2t)$ (d) $t(1+2t)$

5. $L^{-1}\left(\frac{s+2}{s^2-2s+5}\right) =$

   (a) $\cos 2t + \frac{3}{2}\sin 2t$ (b) $\sin 2t + \frac{3}{2}\cos 2t$ (c) $e^t\cos 2t + \frac{3}{2}e^t\sin 2t$ (d) $\cos 2t$

6. $L^{-1}\left(\frac{1}{1-e^{-st}}\int_0^t e^{-at} f(u)du\right) =$

   (a) $f(t)$ (b) $e^{st} f(t)$ (c) $e^{-st} f(t)$ (d) none of the above

7. $L^{-1}\left(\int_s^\infty \bar{f}(s)ds\right) =$

   (a) $\frac{f(t)}{t}$ (b) $\int_0^t f(t)dt$ (c) $\int_0^t \frac{f(t)}{t}dt$ (d) $f(t)$

8. Time domain function of $\frac{s}{s^2+a^2}$ is given by
   a) Cos(at)
   b) Sin(at)
   c) Cos(at)Sin(at)
   d) None of the above
9. If F(s)=L[f(t)], then the formula for $L^{-1}[\int_s^\infty F(s)ds]$ is ______________
10. If F(s)=L[f(t)], then the formulae for (i) $L^{-1}[F'(s)]$ is ________________
11. As per the convolution theorem, $L^{-1}\{\bar{f}(s)\bar{g}(s)\}=$______________________
12. $L^{-1}[\frac{s}{(s+3)^2+4}]=$_________________

13. $L^{-1}\left[\frac{s}{((s)^2+a^2)^2}\right]=$ ____________________

SECTION – B

1. **Find the inverse Laplace transform of** $\frac{s+2}{s^2-4s+13}$
2. Find the inverse Laplace transform of $\frac{2s-5}{4s^2+25}+\frac{4s-18}{9-s^2}$
3. Find the inverse Laplace transform of $\frac{3s+7}{(s^2-2s-3)}$
4. Find the inverse Laplace transform of $\frac{1}{2}log[\frac{s^2+a^2}{s^2+b^2}]$
5. Using convolution theorem to evaluate $L^{-1}[\frac{s^2}{(s^2+a^2)(s^2+b^2)}]$
6. Using convolution theorem to evaluate $L^{-1}[\frac{s^2}{(s^2+a^2)^2}]$
7. Using convolution theorem, evaluate $L^{-1}[\frac{1}{s^2(s+1)^2}]$
8. Find the inverse Laplace theorem of $\frac{1}{s(s+a)(s+b)}$.
9. Solve the differential equation $(D^2+2D+5)y=e^t\sin t;\ y(0)=0,\ y'(0)=1$.
10. Apply "Method of Laplace transforms" ,
    Solve the differential equation $(D^2+2D+5)y=e^t\sin t;\ y(0)=0,\ y'(0)=1$.
11. Apply Laplace transform to the initial value problem $y''+y'-2y=sint$, $y(0)=0, y'(0)=0$.
12. Apply "Method of Laplace transforms" , Solve $x''+2x'+5x=e^t sint, x(0)=0$, $x'(0)=1$.
13. Apply "Method of Laplace transforms" , Solve $x''-3x'+2x=1-e^{2t}, x(0)=1,\ x'(0)=0$.
14. Using Laplace transform, solve $x''+9x=\cos 2t$, if $x(0)=1, x'\left(\frac{\pi}{2}\right)=-1$.
15. Solve, by Laplace transform method, the following initial value problem:

$$(D^2+1)x=tcos2t, such\ that\ x=Dx=0\ at\ t=0$$

<u>GATE PREVIOUS QUESTIONS</u>

1. The function f(t) satisfies the differential equation $\frac{d^2f}{dt^2}+f=0$ and the auxiliary conditions,f(0)=0, $\frac{df}{dt}(0)=4$. The Laplace transform of f(t) is given by **(GATE-2009)**

(a) $\frac{2}{s+1}$ (b) $\frac{4}{s+1}$ (c) $\frac{4}{s^2+1}$ (d) $\frac{2}{s^2+1}$

2. The inverse Laplace transform of the function $F(s)=\frac{1}{s(s+1)}$ is given by

(GATE-2007)

(a) f(t)=sint (b) f(t) = e^{-t} sint (c) e^{-t} (d) 1- e^{-t}

3. The inverse Laplace transform of F(s) = s+1/(s^2+4) is **(GATE-2011)**
(a) cos2t+ sin2t (b) cos2t-(1/2) sin2t (c) cos2t+(1/2)sin2t
(d) cos2t-sin2t

INTEGRAL TRANSFORMS AND MULTIPLE INTEGRALS

Unit – III

FOURIER TRANSFORMS

Objectives:

To introduce

- Fourier transform of a given function and the corresponding inverse.
- Fourier sine and cosine transform of a given function and their corresponding inverses.

Syllabus:

Fourier integral theorem (only statement) – Fourier transform – sine and cosine transforms – properties(with out proofs) – inverse Fourier transforms.

Outcomes:

Students will be able to

- Find Fourier transform and inverse fourier transform of the given function in infinite cases.
- Find Fourier sine and cosine transforms of the given function in infinite cases.

Learning Material

Fourier Transforms are widely used to solve Partial Differential Equations and in various boundary value problems of Engineering such as Vibration of Strings, Conduction of heat, Oscillation of an elastic beam, Transmission lines etc.

Integral Transforms:

- The Integral transform of a function f(x) is defined as

$$I\{f(x)\} = \bar{f}(s) = \int_{x=x_1}^{x_2} f(x)K(s,x)dx$$

Where K(s,x) is a known function of s & x, called the 'Kernel' of the transform.

The function f(x) is called the Inverse transform of $\bar{f}(s)$

1.**Laplace Transform**: When $K(s,x) = e^{-sx}$

$$L\{f(x)\} = \bar{f}(s) = \int_0^\infty f(x)e^{-sx}dx$$

2.**Fourier Transform**: When $K(s,x) = e^{isx}$

$$F\{f(x)\} = \bar{f}(s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)e^{isx}dx$$

3.**Fourier Sine Transform**: When K(s,x)=Sinsx

$$F_s\{f(x)\} = \bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)sinsx\ dx$$

4. **Fourier Cosine Transform**: When K(s,x)=CosSx

$$F_c\{f(x)\} = \bar{f}(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$$

Fourier Integral Theorem:- If f(x) satisfies Dirichlet's conditions for expansion of Fourier series in (-c,c) and $\int_{-\infty}^{\infty}|f(x)|$ converges, then

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt\, d\lambda$$

is known as Fourier Integral of f(x)

Fourier Sine & Cosine Integrals:-

If f(x) satisfies Dirichlet's conditions for expansion of Fourier series in (-c,c) and $\int_{-\infty}^{\infty}|f(x)|$ converges,

- If f(t) is odd function then $f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\, dt\, d\lambda$

is called "Fourier sine Integral" .

- if f(t) is even function then $f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t\, dt\, d\lambda$

This is called "Fourier cosine Integral"

Complex form of Fourier Integral:-

- The complex form of Fourier integral is known as

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)}dt\, d\lambda$$

Problems:

1. using Fourier integral show that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}d\lambda, a,b>0$$

Solution: since the integrand on R.H.S contains sine term, we use Fourier sine integral formula.

We know that fouries sine integral for f(x) is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin px \int_0^\infty f(t)\sin pt dt dp$$

Replacing p with λ we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\int_0^\infty f(t)\sin\lambda t dt d\lambda$$

Here f(x) = $e^{-ax}-e^{-bx}$

f(t) = $e^{-at}-e^{-bt}$

substituting (2) in (1) , we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left(\int_0^\infty (e^{-at}-e^{-bt})\sin\lambda t dt\right)d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\left\{\frac{e^{-at}}{\lambda^2+a^2}(-a\sin\lambda t-\lambda\cos\lambda t)\right\}_0^\infty-\left\{\frac{e^{-bt}}{\lambda^2+b^2}(-b\sin\lambda t-\lambda\cos\lambda t)\right\}_0^\infty\right]d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{\lambda^2+a^2}-\frac{\lambda}{\lambda^2+b^2}\right]d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda(b^2-a^2)}{(\lambda^2+a^2)(\lambda^2+b^2)}\right]d\lambda$$

$$e^{-ax}-e^{-bx}=\frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}d\lambda$$

Fourier Transforms:-

- The fourier transform of a function f(x) is given by F(s) = $\int_{-\infty}^{\infty} f(x)e^{isx}dx$
- The inverse fourier transform of F(S) is given by f(x) = $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$

Fourier Sine transforms:-

- The Fourier sine transform of f(x) is defined as

  $F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$
- The inverse Fourier sine transform of $F_S(S)$ is defined as f(x) = $\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\, ds$

here $F_s(s)$ is called **Fourier sine transform** of **f(x)** and **f(x)** is called **Inverse Fourier sine transform** of $F_s(s)$

Fourier Cosine transforms

- The Fourier cosine transform of f(x) is defined as

  $F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$
- The inverse Fourier cosine transform of $F_S(S)$ is defined as f(x) = $\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\, ds$

here $F_c(s)$ is called **Fourier cosine transform** of **f(x)** and **f(x)** is called **Inverse Fourier cosine transform** of $F_c(s)$

NOTE: 1. Some authors define F.T as follows

i) F(s) = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ist}dt$ ii) f(x) = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$

iii) F(s) = $\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx$ iv) f(x) = $\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} F(s)e^{isx}ds$

2.Some authors define Fourier sine & cosine transforms as follows

i) $F_s(s) = \int_0^\infty f(x)\sin sx\, dx$ ii) f(x) = $\frac{2}{\pi}\int_0^\infty F_s(s)\sin sx\, ds$

iii) $F_c(s) = \int_0^\infty f(x)\cos sx\, dx$ iv) f(x) = $\frac{2}{\pi}\int_0^\infty F_c(s)\cos sx\, ds$

Properties of Fourier Transforms:-

1. **Linearity Property**:- If $F_1(s)$ and $F_2(s)$ be the Fourier transforms of $f_1(x)$ and $f_2(x)$
2. respectively then f{$af_1(x) + b\, f_2(x)\} = a$ $F_1(s) + b\, F_2(s)$, where a & b are constants
3. .**Change of Scale Property**:- If F{f(x)} = F(s) then F{f(ax)} = $\frac{1}{a} F(\frac{s}{a})$
4. **Shifting Property**:- If F{f(x)} = F(s) then F{f(x-a)} = $e^{isa} F(s)$
5. **Modulation Property**:- If F{f(x)} = F(s) then F{f(x)cosax} = ½ {F(s+a)+F(s-a)}
6. If F{f(x)} = F(s) then F{f(-x)}= F(-s)
7. $\overline{F\{f(x)\}} = \overline{F(-s)}$
8. $\overline{F\{f(-x)\}} = \overline{F(s)}$
9. $F_c\{xf(x)\} = \frac{d}{ds} F_s\{f(x)\}$

Problem : Derive the relation between Fourier transform and Laplace transform.

Solution: consider $f(t) = \begin{cases} e^{-xt} g(t), t > 0,, \\ 0, t < 0 \end{cases}$

The fourier trasform of f(x) is given by

$$F(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-xt} g(t) e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{(is-x)t} g(t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-pt} g(t) dt \quad \textbf{where p=x-is}$$

$$= \frac{1}{\sqrt{2\pi}} L(g(t)) \because \left[L\left(f(t) = \int_0^{\infty} e^{-st} f(t) dt \right) \right]$$

$\because$ **Fourier transform of** $f(t) = \frac{1}{\sqrt{2\pi}} \times$ **laplace transform of g(t)**

Problems:

Find the F.T of f(x) = $e^{-|x|}$

sol: Given f(x) = $e^{-|x|}$

$$= \begin{cases} e^{x}; x < 0 \\ e^{-x}; x > 0 \end{cases}$$

by definition, F{f(x)} = $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \{ \int_{-\infty}^{0} f(x) e^{isx} dx + \int_0^{\infty} f(x) e^{isx} dx \}$$

$$= \frac{1}{\sqrt{2\pi}} \{ \int_{-\infty}^{0} e^{(1+is)x} dx + \int_0^{\infty} e^{(-1+is)x} dx \}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left(\frac{e^{(1+is)x}}{1+is} \right)_{-\infty}^{0} + \left(\frac{-e^{-(1-is)x}}{1-is} \right)_{0}^{\infty} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+is} + \frac{1}{1-is} \right)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2}$$

Problem: Find the Fourier transform of f(x) defined by $f(x) = \begin{cases} 1, \text{ if } |x| < a \\ 0, \text{ if } |x| > a \end{cases}$

And hence evaluate $\int_0^\infty \frac{\sin p}{p} dp$ and $\int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p} dp$

Sol: We have F[f(x)]= $\int_{-\infty}^{\infty} e^{ipx} f(x)dx = \int_{-\infty}^{-a} e^{ipx} f(x)dx + \int_{-a}^{a} e^{ipx} f(x)dx + \int_{a}^{\infty} e^{ipx} f(x)dx$

$$= \int_{-a}^{a} e^{ipx} dx = \frac{2\sin ap}{p}$$

By the inversion formula, we know that f(x)= $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} F(p)dp$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \frac{2\sin ap}{p} dp = \begin{cases} 1, \text{ if } |x| < a \\ 0, \text{ if } |x| > a \end{cases}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \cos px \frac{2\sin ap}{p} dp - \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin px \frac{2\sin ap}{p} dp = \begin{cases} 1, \text{ if } |x| < a \\ 0, \text{ if } |x| > a \end{cases}$$

Since the second integral is an odd function, $\int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p} dp = \pi ,$

$\begin{cases} 1, \text{ if } |x| < a \\ 0, \text{ if } |x| > a \end{cases}$

Put x=0, we get, $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin ap}{p} dp = \begin{cases} 1, \text{ if } a > 0 \\ 0, \text{ if } a < 0 \end{cases}$

$$\int_0^\infty \frac{\sin p}{p} dp = \frac{\pi}{2}, \quad a>0$$

$$= 0, \quad a<0$$

And put x=0 and a=1 then we get $\int_0^\infty \frac{\sin p}{p} dp = \frac{\pi}{2}$

Problem: Find the fourier sine transform of e^{-ax}, a>0 and hence deduce that $\int_0^\infty \frac{p \sin px}{a^2 + p^2} dp$

Sol: $F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx = \int_0^\infty e^{-ax}\sin px\,dx = \dfrac{p}{a^2+p^2}$

By the inversion formula, we know that f(x)= $\dfrac{2}{\pi}\int_0^\infty F_s\{f(x)\}\sin px\,dp$

$$= \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp$$

$$\therefore \int_0^\infty \frac{p\sin px}{a^2+p^2}\,dp = \frac{\pi}{2}e^{-ax}$$

Problem : Find the Fourier sine Transform of $\frac{1}{x}$.

Sol: $F_s\{f(x)\} = \int_0^\infty f(x)\sin px\,dx = \int_0^\infty \frac{1}{x}\sin px\,dx$

Let px=θ

Pdx=dθ , θ:0→∞

$$F_s\{f(x)\} = \int_0^\infty \frac{p}{\theta}\sin\theta\,\frac{d\theta}{p}$$

$$= \sqrt{\frac{\pi}{2}}$$

Problem : Find the Fourier cosine transform of f(x) = $\begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Solution : $F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx = \int_0^a \cos x\cos px\,dx$

$$= \int_0^a \frac{\cos(p+1)x + \cos(p-1)x}{2}\,dx$$

$$= \left[\frac{\sin(p+1)a}{p+1} + \frac{\sin(p-1)a}{p-1}\right]\frac{1}{2}$$

Assignment-Cum-Tutorial Questions

SECTION-A

Objective / Multiple choice Questions:

1. Fourier Cosine transform of f(x) is ____________________.
2. Fourier sine transform of f(x) is ____________________.
3. Fourier transform of f(x) is ____________________.
4. $\int_0^\infty e^{-ax} \sin bx \, dx =$ ____________________
5. Fourier Cosine transform of f(x) = e^{-ax} is ____________________.
6. Fourier sine transform of f(x) = e^{-ax} is ____________________.
7. $\int_0^\infty e^{-ax} \cos bx \, dx =$ ____________________
8. The inverse Fourier cosine transform of $f(x)$ is

   (a) $\sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$ (b) $\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty F_c(s) \cos sx \, dx$

   (c) $\sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos s \, dx$ (d) None.

9. $\int_0^\infty e^{-x} \cos 3x \, dx =$ ____________________
10. If $\tilde{f}(\alpha)$ is the Fourier transform of f(x), then the Fourier Transform of f(t-a) is

    a) $e^{i\alpha a} \tilde{f}(\alpha)$ b) $e^{i\alpha a}$ c) $e^{-i\alpha a} \tilde{f}(\alpha)$ d) $\frac{1}{a} \tilde{f}(\alpha / a)$

11. If $F\{f(x)\} = \tilde{f}(\alpha)$ then $F\{f(ax)\}$ is

    a) $\frac{1}{a} \tilde{f}(\alpha / a)$ b) $\tilde{f}(\alpha / a)$ c) $af(ax)$ d) None of these

12. If $F\{f(x)\} = \tilde{f}(\alpha)$ then $F\{f^{(n)}(x)\} =$ ....

    a) $\alpha^n \tilde{f}(\alpha)$ b) $(i\alpha)^n \tilde{f}(\alpha)$ c) $i^n \tilde{f}(\alpha / n)$ d) None of these

13. If $F\{e^{-|x|}\} = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2}$ then the value of the Fourier transform of $5e^{-4|x+2|}$ is

a) $\sqrt{\frac{2}{\pi}}.20e^{2i\alpha}\frac{1}{16+\alpha^2}$ b) $e^{2i\alpha}\frac{1}{16+\alpha^2}$ c) $\sqrt{\frac{2}{\pi}}\frac{20}{16+\alpha^2}$ d) None of these

SECTION-B

II) *Descriptive Questions*:

1. Find the Fourier transform of $f(x)=\begin{cases} e^{ikx} & a<x<b \\ 0 & x<a,\ x>b \end{cases}$.
2. Find the Fourier transform of $f(x)=\begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$, hence evaluate $\int_0^\infty \frac{S\text{int}}{t}dt$.
3. Find the Fourier transform of $f(x)=e^{-x^2/2}, -\infty<x<\infty$ [or] S.T Fourier transform of $e^{-x^2/2}$ is self reciprocal.
4. Find the Fourier transform of f(x) defined by $f(x)=\begin{cases} a^2-x^2, if\ |x|\le 1 \\ 0, \quad if\ |x|>1 \end{cases}$.

   And S.T. $\int_0^\infty \frac{\sin t - t\cos t}{t^3}dt=\frac{\pi}{4}$
5. Find the Fourier cosine and sine transform of $5e^{-2x}+2e^{-5x}$
6. Find the **a)** Fourier cosine and **b)** Fourier Sine transform of f(x) = e^{-ax} for x≥0 and a>0. And hence deduce the integrals known as "Laplace integrals" $\int_0^\infty \frac{\cos\alpha x}{\alpha^2+a^2}d\alpha$ and $\int_0^\infty \frac{\alpha.Sin\alpha x}{\alpha^2+a^2}d\alpha$
7. Find the inverse Fourier cosine transform f(x) if

   $$F_c(\alpha)=\begin{cases} \frac{1}{2a}(a-\frac{\alpha}{2}), & \alpha<2a \\ 0, & \alpha\ge 2a \end{cases}$$
8. Find Fourier sine transform f(x)=$e^{-|x|}$ & hence find $\int_0^\infty \frac{x.\sin mx}{1+x^2}dx$
9. Find the Fourier cosine and sine transform of xe^{-ax}.
10. Find the Fourier sine and cosine transforms of $f(x)=\frac{e^{-ax}}{x}$. S.T.

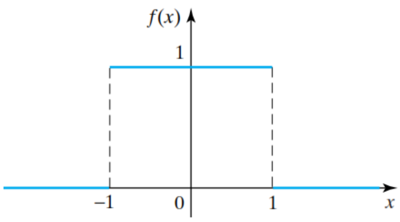
    $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}Sinsxdx=\tan^{-1}\left(\frac{s}{a}\right)-\tan^{-1}\left(\frac{s}{b}\right).$$

SECTION-C

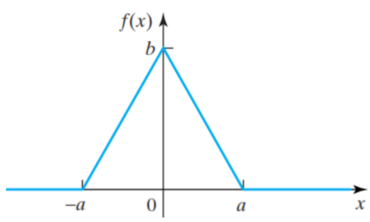
C. Questions testing the analyzing / evaluating ability of students

Find the Fourier integral representation of the following functions.

1. The rectangular pulse function



2. The triangular function



Hint :

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Hint

$$f(x) = \begin{cases} 0, & |x| > a \\ b\left(1+\frac{x}{a}\right), & -a \le x \le 0 \\ b\left(1-\frac{x}{a}\right), & 0 \le x \le a \end{cases}$$

GATE PREVIOUS QUESTIONS

The value of the integral

$$\int_{-\infty}^{\infty} \sin c^2(dt) \quad is \_\_\_\_\_\_\_.$$

[**2014**]

ANS. 0.2

. Let $g(t) = e^{-\pi t^2}$, and $h(t)$ is filter marched to $g(t)$. If $g(t)$ is applied as input to $h(t)$, then the Fourier transform of the output is

(a) $e^{-\pi t^2}$ (c) $e^{-\pi|f|}$

(b) $e^{-\pi f^2/2}$ (d) $e^{-2\pi f^2}$ **[2013]**

ANS. (D)

.The Fourier transform of a signal h(t) is $H(j\omega) = (2\cos\omega)(\sin 2\omega)/\omega$. The value of h(0) is

(a) 1/4 (c) 1

(b) 1/2 (d) 2 **[2012]**

ANS. (C)

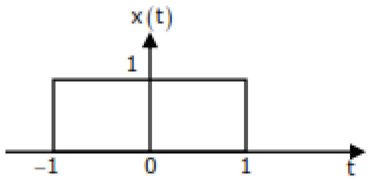
x(t) is a positive rectangular pulse from *t* = -1 to *t* = +1 with unit height as shown in the figure. The value of $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$ {where $X(\omega)$ is the Fourier transform of x(t)} is

(A) 2

(B) 2π

(C) 4

(D) 4π



[2010]

INTEGRAL TRANSFORMS AND MULTIPLE INTEGRALS

Unit – IV

MULTIPLE INTEGRALS (only Cartesian form)

Objectives:

- To know about double and triple integrals
- Understand the use of double and triple integrals in finding areas and volumes

Syllabus:

Double integrals - areas, triple integrals - Volume.

Out comes:

At the end of this topic, students are able to:

- Compute the double and triple integrals
- Apply double and triple integral concept to find areas and volumes

MULTIPLE INTEGRALS

DOUBLE INTEGRAL

Evaluation Of Double Integrals in Cartesian coordinates

- If all the four limits are constats ,then the double integral can be evaluated in either way. i.e. we first integrate w.r.t .x and then w.r.t y or we first integrate w.r.t .y and later w.r.t x.

Example (1) : $\int_0^2\int_0^3 xy\,dx\,dy$

Sol: Here all the four limits are constants. So the double integral can be evaluated in either way i.e. we first integrate w.r.t .x and then w.r.t y or we first integrate w.r.t .y and later w.r.t x.

$$\int_0^2\int_0^3 xy\,dx\,dy = \int_0^2\left[\frac{x^2y}{2}\right]_0^3 dy = \frac{9}{2}\int_0^2 y\,dy = 9$$

- Suppose that region R can be described by inequalities' of the form $a\le x\le b$, $y_1(x)\le y_2(x)$ so that $y=y_1(x)$ and $y=y_2(x)$ represent the boundary of R. Then the Double Integral over the region R may be evaluated as follows.

$$\int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx = \int_{x=a}^{b}\left[\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\right]dx$$

Example (2) :Evaluate $\int_0^5\int_0^{x^2} x\left(x^2+y^2\right)dx\,dy$

Sol: $\int_0^5\int_0^{x^2} x(x^2+y^2)dxdy = \int_0^5\left[\int_0^{x^2} x(x^2+y^2)dy\right]dx$

$$= \int_0^5\left(x^3y+\frac{xy^3}{3}\right)_0^{x^2} dx$$

$$= \int_0^5\left(x^5+\frac{x^7}{3}\right)dx$$

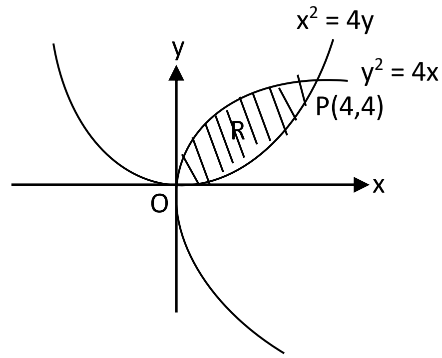
$$= \left(\frac{x^6}{6}+\frac{x^8}{24}\right)_0^5 = \frac{29(5)^6}{24}$$

Example(3) : Evaluate $\iint_R ydxdy$ Where R is the region bounded by the parabolas $y^2=4x$ and $x^2=4y$

Sol: Given parabolas $y^2 = 4ax$ ....(1)

and $x^2 = 4y$

are shown in figure.



To find their points of intersection ,solve (1) and (2).

Substituting the values of y from (2) in (1),we get

$$\left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow x^4 = 4^3x \Rightarrow x(x^3-4^3)=0$$

x=0 or x=4

Thus the two parabolas intersect at the points O(0,0) and P(4,4)

The shaded area between the parabolas (1) and (2) is the region of integration

$$\iint_R ydxdy = \left[\int_0^4\left[\int_{\frac{x^2}{4}}^{2\sqrt{x}} ydy\right]dx\right]$$

$$= \left[\int_0^4\left[\frac{y^2}{2}\right]_{\frac{x^2}{4}}^{2\sqrt{x}} dx\right]$$

$$= \left[\int_0^4\left(4x-\frac{x^4}{16}\right)dx\right]$$

$$= \frac{1}{2}\left[2x^2-\frac{x^5}{80}\right]_0^4$$

$$= \frac{48}{5}$$

- Suppose that region R can be described by inequalities' of the form $a \le y \le b$, $x_1(y) \le x_2(y)$ so that $x = x_1(x)$ and $x = x_2(x)$ represent the boundary of R. Then the Double Integral over the region R may be evaluated as follows.

$$\int_{y=a}^{b}\int_{x=x_1(y)}^{x_2(y)} f(x,y)dxdy = \int_{y=a}^{b}\left[\int_{x=x_1(y)}^{x_2(y)} f(x,y)dx\right]dy$$

Example(4): Evaluate $\int_0^4 \int_{\frac{y^2}{4}}^{y} \frac{y}{x^2+y^2}dxdy$

$$\int_0^4 \int_{\frac{y^2}{4}}^{y} \frac{y}{x^2+y^2}dxdy = \int_0^4 \left[\int_{\frac{y^2}{4}}^{y} \frac{y}{x^2+y^2}dx\right]dy$$

$$= \int_0^4 \left[y\left\{\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)\right\}_{\frac{y^2}{4}}^{y}\right]dy$$

$$= \int_0^4 \left[\tan^{-1}(1) - \tan^{-1}\left(\frac{y}{4}\right)\right]dy$$

$$= \int_0^4 \left[\frac{\pi}{4} - \tan^{-1}\left(\frac{y}{4}\right)\right]dy$$

$$= \frac{\pi}{4}(y)_0^4 - \int_0^4 \tan^{-1}\frac{y}{4}dy$$

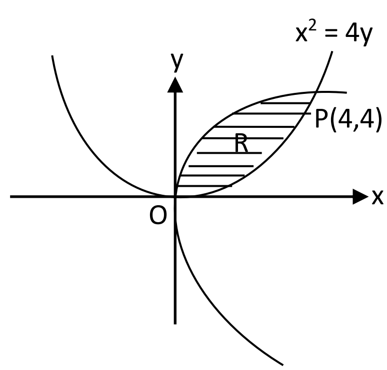
$$= \pi - \left[\left\{\tan^{-1}\left(\frac{y}{4}\right).y\right\}_0^4 - \int_0^4 y.\frac{1}{1+\frac{y^2}{16}}\frac{1}{4}dy\right]$$

$$= \pi - \left[\left\{4\tan^{-1}(1)\right\} - 4\int_0^4 y.\frac{1}{16+y^2}dy\right]$$

$$= \pi - \left[4\left(\frac{\pi}{4}\right) - 2\left\{\log\left(y^2+16\right)\right\}_0^4\right]$$

$$= 2[\log 32 - \log 16] = 2\log 2$$

Example(5): Evaluate $\iint_R ydxdy$ Where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$



the two parabolas intersect at the points O(0,0) and P(4,4)

The shaded area between the parabolas (1) and (2) is the region of integration

$$\iint_R y\,dx\,dy = \left[\int_0^4 y\left[\int_{\frac{y^2}{4}}^{2\sqrt{y}} dx\right]dy\right]$$

$$= \left[\int_0^4 y\left[-\frac{y^2}{4}+2\sqrt{y}\right]dy\right]$$

$$= \left[\int_0^4 \left(2y^{\frac{3}{2}}-\frac{y^3}{4}\right)dy\right]$$

$$= \left[\frac{4}{5}y^{\frac{5}{2}}-\frac{y^4}{16}\right]_0^4$$

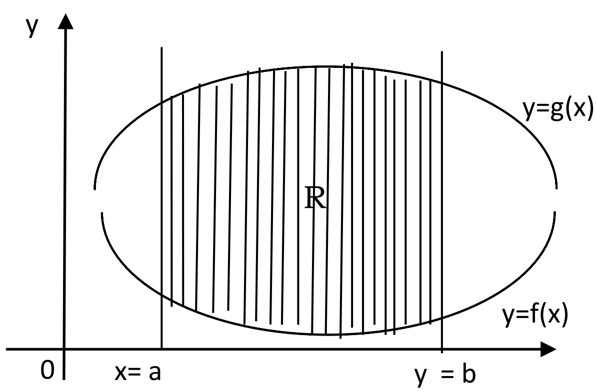
$$\frac{48}{5}$$

- **<u>Area enclosed by a plane curve (Cartesian coordinates)</u>**

Consider the area enclosed by the curves y=f(x),y=g(x),x=a,x=b in the xy plane.

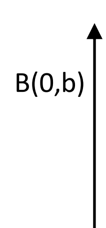
Then area of the region R bounded by the curves is given by $\iint_R dxdy$ or

$$\iint_R dydx = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} dydx$$



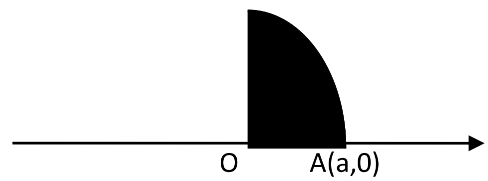
Example(8) :Find the area of a plate in the form of a quadrant of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Slotion: Limits of y are: 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

Limits of x are :0 to a



$$\text{Area} = \int_0^a \left[\int_0^{\frac{b\sqrt{a^2-x^2}}{a}} dy \right] dx$$

$$= \int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx$$

$$= \frac{b}{a}\left[\left[\frac{x}{2}\sqrt{a^2-x^2}\right] + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{\pi ba}{4} \text{ square units.}$$

➢ Triple Integrals:

Integration of a function $f(x,y,z)$ over a three dimensional region V is called the triple integral.

i.e., $\iiint\limits_V f(x,y,z)\,dV = \int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}\int\limits_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$ .... (1)

Like double integrals, in triple integrals also the order of integration depends on the nature of the limits of variables.

Problem: Evaluate $\int_0^1\int_y^1\int_0^{1-x} x\,dz\,dx\,dy$

Sol: $\int_0^1\int_y^1\int_0^{1-x} x\,dz\,dx\,dy = \int_0^1\left[\int_y^1 x(z)_0^{1-x}\,dx\right]dy$

$$= \int_0^1\left[\int_y^1 x(1-x)\,dx\right]dy = \int_0^1\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_y^1 dy$$

$$= \int_0^1\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{y^2}{2}-\frac{y^3}{3}\right)dy = \int_0^1 \frac{1}{6}-\left(\frac{y^2}{2}-\frac{y^3}{3}\right)dy = \frac{1}{12}$$

Applications of Triple Integrals:

1. ***Volume as a triple integral:***

If $f(x,y,z) = 1$ then the value of equation (1) is called volume V of the region R.

$$\text{i.e.,} \iiint_V dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dx\,dy\,dz$$

Assignment-Cum-Tutorial Questions

SECTION-A

Objective Questions :

1) The value of $\int_0^{\sqrt{3}}\int_{-\sqrt{9-3y^2}}^{\sqrt{9-3y^2}} y\,dx\,dy$ is ________

2) $\int_0^{\log t}\int_0^{x} e^{x+y}\,dx\,dy =$ ________________

3) $\int_1^{p}\int_2^{q} \frac{dx\,dy}{x\,y} =$__________

4) $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$ is ___________

5) $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z}(x+y+z)dy\,dx\,dz$ is _______________

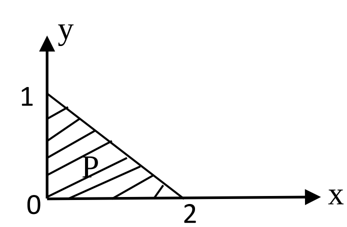
6) $\int_0^1\int_0^2\int_1^2 x\,y\,z\,dx\,dy\,dz =$ ____________

7) The form of area bounded by the parabola $y^2 = x$ and the line $x - y = 2$ can be expressed as_______________

8)The double integral $\iint_{D_a} e^{-(x^2+y^2)}\,dA$ where D_a is the disc of radius $a > 0$ centered at the origin, is [ ]

a) $2\pi e^{-a^2}$ b) $\pi\left(1-e^{-a^2}\right)$ c) 0 d) $\pi\left(e^{-a^2}-1\right)$

9) Consider the shaded triangular region P shown in fig., then $\iint_P dx\,dy$ is _______ [ ]



a) $\frac{2}{3}$ b) $\frac{1}{6}$ c) $\frac{1}{2}$ d) $\frac{1}{3}$

10) Area bounded by the curve $y^2 = x$ and the line $x=3$ is _________sq. units [ ]

a) $2\sqrt{3}$ b) $4\sqrt{3}$ c) $6\sqrt{3}$ d) $8\sqrt{3}$

11) Area bounded by the curve $y = -3x^2$, $x=2$ and the two coordinate axes is _________sq. units [ ]

.

a) 2 b) 3 c) 6 d) 8

12) If E is the solid region bounded by the co-ordinate planes and $2x+2y+z=4$ then the $\iiint_E y\,dv$ is _________ [ ]

a) -4/3 b) 4 c) 0 d) 4/3

SECTION-B

II) *Descriptive Questions*:

1) Find $\int_0^4\int_0^{x^2} e^{y/x}\, dydx$

2) Find $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dydx$

3) Find $\int_0^1\int_0^1\int_{\sqrt{x^2+y^2}}^2 xyz\,dzdydx$

4) Find $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_o^{\sqrt{x^2+y^2}} z^2\, dzdydx$

5) Evaluate $\int_1^e\int_1^{\log y}\int_1^{e^x} \log z\,dzdydx$.

SECTION-C

C. Questions testing the analyzing / evaluating ability of students

1) If R is the triangular region bounded by the coordinate axis and the line $\frac{x}{a}+\frac{y}{b}=1$, find $\iint_R y\,dxdy$

2) Find $\iint_R f(x,y)dR$ where $f(x,y)=x^2+y^2$, R is the region in the first quadrant bounded by the coordinate axis and $x+y\le 1$

3) Evaluate $\iint_R (x+y)^2\, dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

4) Find the area of the region bounded by x-axis and the curves $y^2= 4ax$, $x+y=3a$

5) Find the area enclosed by the curves $y=4x-x^2$, $y=x$

6) Find $\iiint_R z\,dxdydz$ where R is the region inside the upper part of the sphere $x^2+y^2+z^2=a^2$ and the xy- plane

7) Evaluate $\iiint xyz\,dydxdz$ over the domain bounded by the coordinate planes and the plane $x+y+z=1$

8))Find the volume bounded by the cylinder $x=9-y^2$ and the planes $z=y,\ x=0, z=0$ in

the first octant .

GATE PREVIOUS QUESTIONS

1) To evaluate the double integral $\int_0^8 \left(\int_{\frac{y}{2}}^{\left(\frac{y}{2}\right)+1} \left(\frac{2x-y}{2}\right) dx\right) dy$, we make the substitution u = $\left(\frac{2x-y}{2}\right)$ and v = $\frac{y}{2}$ The integral will reduce to **[2014]**

a) $\int_0^4 \left(\int_0^2 2u\,du\right) dv$ b) $\int_0^4 \left(\int_0^1 2u\,du\right) dv$ c) $\int_0^4 \left(\int_0^1 u\,du\right) dv$ d) $\int_0^4 \left(\int_0^2 u\,du\right) dv$

2) f(x,y) is a continuous function defined over (x,y) ∈ [0,1] x [0,1]. Given the two constraints

$x > y^2$ and $y > x^2$, the volume under f(x,y) is
[2009]

a) $\int_{y=0}^{y=1} \int_{x=y^2}^{x=\sqrt{y}} f(x,y)dx\,dy$ b) $\int_{y=x^2}^{y=1} \int_{x=y^2}^{x=1} f(x,y)dx\,dy$

c) $\int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x,y)dx\,dy$ d) $\int_{y=0}^{y=\sqrt{x}} \int_{x=0}^{x=\sqrt{y}} f(x,y)dx\,dy$

INTEGRAL TRANSFORMS AND MULTIPLE INTEGRALS

Learning Material

UNIT V: Vector Differentiation

INTRODUCTION

Course Objectives:

- To introduce concept of gradient
- To introduce concept of divergence and
- To introduce concept of curl

Course Outcomes:

- To understand the physical interpretations of gradient
- To apply the gradient in various physical and engineering problems.
- To understand the physical interpretations of divergence
- To understand the physical interpretations of curl

Learning Material

❖ **UNIT V: Vector Differentiation**

❖ **Scalar and vector point functions:** Consider a region in three dimensional space. To each point $p(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a ***scalar point function***. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$we associate a unique vector $\bar{f}\,(x,y,z)$. $\bar{f}$ is called a ***vector point function.***

Examples:

1. For example take a heated solid. At each point p(x,y,z)of the solid, there will be temperature $T(x,y,z)$. This T is a scalar point function.
2. Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position $p(x,y,z)$ in space, it will be having some speed, say, v. This **speed** v is a scalar point function.
3. Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.
4. In a magnetic field, at any point P(x,y,z) there will be a magnetic force $\bar{f}\,(x,y,z)$. This is called magnetic force field. This is also an example of a vector point

function. The students will come across several scalar and vector point functions in their respective subjects of study.

❖ **Tangent vector to a curve in space.**

Consider an interval [a,b].

Let x = x(t),y=y(t),z=z(t)be continuous and derivable for a≤ t ≤b.

Then the set of all points (x(t),y(t),z(t)) is called a curve in a space.

Let A = (x(a),y(a),z(a)) and B = (x(b),y(b),z(b)).

Then A, B are called the end points of the curve and if A =B, the curve in said to be a closed curve.

If $\bar{r} = (x(t), y(t), z(t))$ at the point P on the curve $(i.e., \bar{r} = \overline{OP})$ then $\frac{d\bar{r}}{dt}$ will be a tangent vector to the curve at P. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector and Suppose arc length AP = s. if we take the parameter as the arc length , we can observe that $\frac{d\bar{r}}{ds}$ is unit tangent vector at P to the curve.)

❖ **<u>VECTOR DIFFERENTIAL OPERATOR</u>**

Def. The vector differential operator ∇(read as del) is defined as $\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$.

This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator.

We will define now some quantities known as "**gradient", "divergence"** and **"curl"** involving the operator ∇.

We must note that this operator has no meaning by itself unless it operates on some function suitably.

<u>GRADIENT OF A SCALAR POINT FUNCTION</u>

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ and is denoted by gradϕ or $\nabla\phi$.

i.e., $\nabla\phi = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$

Properties:

(1) If f and g are two scalar functions then grad(f ±g)= grad f ± grad g

(2) The necessary and sufficient condition for a scalar point function 'f' to be constant is that $\nabla f = \bar{O}$

(3) grad(fg) = f (grad g)+g(grad f)

(4) If c is a constant, grad (cf) = c(grad f)

(5) grad $\left(\frac{f}{g}\right) = \frac{g(grad\ f) - f(grad\ g)}{g^2}$, $(g \neq 0)$

(6) Let r = $x\bar{i}+y\bar{j}+z\bar{k}$. Then $d\bar{r}$= $(dx)\bar{i}+(dy)\bar{j}+(dz)\bar{k}$. if ϕ is any scalar point function,

then $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(\bar{i}dx + \bar{j}dy + \bar{k}dz) = \nabla \phi . d\bar{r}$

DIRECTIONAL DERIVATIVE

Let ϕ(x,y,z) be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is OP = r. Then the directional derivative of ϕ at P and is denoted by $d\phi/dr$.

Result 1: *The directional derivative of a scalar point function ϕ at a point P(x,y,z) in the direction of a unit vector* $\bar{e}$ *is equal to* $\bar{e}$. grad ϕ=$\bar{e}$. $\nabla\phi$.

Level Surface

If a surface ϕ(x,y,z)= c be drawn through any point P($\bar{r}$), such that at each point on it, function has the same value as at P, then such a surface is called a level surface of the function ϕ through P.

e.g : equipotential or isothermal surface.

Result 2: *$\nabla\phi$ at any point is a vector normal to the level surface ϕ(x,y,z)=c through the point, where c is a constant.*

Result 3: The physical interpretation of $\nabla\phi$

The gradient of a scalar function ϕ(x,y,z) at a point *P*(x,y,z) is a **vector along the normal** to the level surface ϕ(x,y,z) = c at *P* and is in increasing direction. Its magnitude is equal to the greatest rate of increase of ϕ . *The Greatest value of directional derivative of* ***ϕ at a point P = |grad ϕ|*** *at the point.*

Example 1: Find the directional derivative of $f = xy+yz+zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).

Sol:- Given $f = xy+yz+zx$.

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{z}\frac{\partial f}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$, then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

Directional derivative of f along the given direction $= \bar{e}.\nabla f$

$= \frac{1}{3}(i + 2j + 2k).[(y+z)i + (z+x)j + (x+y)k]\ at\ (1,2,0)$

$= \frac{1}{3}[(y+z) + 2(z+x) + 2(x+y)]\ at\ (1,2,0) = \frac{10}{3}$

Example 2: Show that $\nabla[f(r)] = \frac{f'(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol:-since $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, we have $r^2 = x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly\ \frac{\partial r}{\partial y} = \frac{y}{r}\ ,\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla[f(r)] = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(r) = \sum \bar{i} f'(r)\frac{\partial r}{\partial x} = \sum \bar{i} f'(r)\frac{x}{r}$

$= \frac{f'(r)}{r}\sum \bar{i}x = \frac{f'(r)}{r}\bar{r}$

Note : From the above result, $\nabla(\log r) = \frac{1}{r^2}\bar{r}$

Example 3: Find a unit normal vector to the given surface $x^2y+2xz = 4$ at the point (2,-2,3).

Sol:- Let the given surface be $f = x^2y+2xz - 4$

On differentiating,

$$\frac{\partial f}{\partial x} = 2xy + 2z,\ \frac{\partial f}{\partial y} = x^2, \frac{\partial f}{\partial z} = 2x.$$

grad $f = \sum \bar{i}\frac{\partial f}{\partial x} = \bar{i}(2xy+2z) + \bar{j}x^2 + 2x\bar{k})$

(grad f) at (2,-2,3) = $\bar{i}(-8+6) + 4\bar{j} + 4\bar{k}) = -2\bar{i} + 4\bar{j} + 4\bar{k}$

grad (f) is the normal vector to the given surface at the given point.

Hence the required unit normal vector is $\frac{\nabla f}{|\nabla f|} = \frac{2(-\bar{i} + 2\bar{j} + 2\bar{k}).}{2\sqrt{1+2^2+2^2}} = \frac{-\bar{i} + 2\bar{j} + 2\bar{k}}{3}$

Example 4: Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at (2,1,-1).

Sol: we have

grad $f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = -4\bar{i} - 4\bar{j} + 12\bar{k}$ at (2,1,-1).

Greatest value of the directional derivative of $f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$

Example 5: If $\bar{a}$ is constant vector then prove that grad $(\bar{a}\,.\,\bar{r}) = \bar{a}$

Sol: Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, where a_1, a_2, a_3 are constants.

$\bar{a}\,.\,\bar{r} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}).(x\bar{i} + y\bar{j} + z\bar{k}) = a_1x + a_2y + a_3z$

$\frac{\partial}{\partial x}(\bar{a}.\bar{r}) = a_1, \frac{\partial}{\partial y}(\bar{a}.\bar{r}) = a_2, \frac{\partial}{\partial z}(\bar{a}.\bar{r}) = a_3$

grad $(\bar{a}\,.\,\bar{r}) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

Example 6: If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find ϕ.

Sol:- we know that $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Given that $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$

Comparing the corresponding coefficients, we have $\frac{\partial\phi}{\partial x} = yz, \frac{\partial\phi}{\partial y} = zx, \frac{\partial\phi}{\partial z} = xy$

Integrating partially w.r.t. x,y,z, respectively, we get

ϕ= xyz + a constant independent of x.

ϕ= xyz + a constant independent of y.

ϕ= xyz + a constant independent of z.

Here a possible form of ϕ is ϕ= xyz+a constant.

DIVERGENCE OF A VECTOR

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e., div $\bar{f} = \bar{i}.\frac{\partial\bar{f}}{\partial x} + \bar{j}.\frac{\partial\bar{f}}{\partial y} + \bar{k}.\frac{\partial\bar{f}}{\partial z} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

hence we can write div $\bar{f}$ as div $\bar{f}$ = $\nabla.\,\bar{f}$

This is a scalar point function.

Result 1: If the vector $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Result 2: div $(\bar{f} \pm \bar{g}) = div\,\bar{f} \pm div\,\bar{g}$

Result 3: A vector point function $\bar{f}$ is said to be $\bar{f}$ ***solenoidal*** if div $\bar{f}$ =0.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem like fluid dynamics, electricity and magnetism etc, we can interpret div $\bar{f}$ (= $\nabla.\,\bar{f}$).

1. Suppose $\bar{f}$ (x,y,z) is the velocity of a fluid at a point (x,y,z). Imagine a small rectangular box within the fluid. The divergence of $\bar{f}$ gives the rate at which the fluid flows out per unit volume at any given time. Therefore divergence of $\bar{f}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

If div $\bar{f}$ =0, then the fluid entering and leaving the element is the same, i.e., there is no change in the density of the fluid (or fluid is incompressible.)

2. The divergence of current density **J** gives the amount of charge flowing out per unit volume per second from a small element of closed surface around that point.

If div**J** =0 then it shows that the medium is free of charges.

Example 1: If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

Sol:- $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$. Then

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2) = y^2+2x^2z-6yz$

(div $\bar{f}$) at (1, -1, 1) = 1+2+6 =9

Example 2: find div $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$. Then

$$\frac{\partial \phi}{\partial x} = 3x^2 - 3yz, \frac{\partial \phi}{\partial y} = 3y^2 - 3zx, \frac{\partial \phi}{\partial z} = 3z^2 - 3xy$$

$$\text{grad } \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$$

$$\text{div } \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)] + \frac{\partial}{\partial y}[3(y^2 - zx)] + \frac{\partial}{\partial z}[3(z^2 - xy)]$$

= 3(2x)+3(2y)+3(2z) = 6(x+y+z)

Example 3: If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is ***solenoidal***, find *P*.

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1, \frac{\partial f_2}{\partial y} = 1, \frac{\partial f_3}{\partial z} = p$

$$\text{div } \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 1+1+p = 2+p$$

since $\bar{f}$ is solenoidal, we have div $\bar{f} = 0$. Hence p = -2.

Example 4: Find div $\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{div } \bar{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function.

Then the vector function defined by $\bar{i}X\frac{\partial \bar{f}}{\partial x} + \bar{j}X\frac{\partial \bar{f}}{\partial y} + \bar{k}X\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or ∇x $\bar{f}$.

$$\text{Curl } \bar{f} = \bar{i}X\frac{\partial \bar{f}}{\partial x} + \bar{j}X\frac{\partial \bar{f}}{\partial y} + \bar{k}X\frac{\partial \bar{f}}{\partial z} = \sum\left(\bar{i}X\frac{\partial \bar{f}}{\partial x}\right)$$

Result 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then curl $\bar{f}$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\textit{i.e.,} \; curl\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Result 2: A vector $\bar{f}$ is said to be ***irrotational*** vector if curl $\bar{f} = \bar{o}$.

Physical Interpretation of curl

The general meaning of curl is rotation. When curl $\bar{f} = \bar{o}$, it means that no rotation is attached with the vector $\bar{f}$ whereas if curl $\bar{f}$ is non zero, it means that rotation is attached with the vector $\bar{f}$.

If $\bar{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point P(x,y,z) on the body, then $\bar{w}$ = ½ curl $\bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. Hence "***curl*** of a vector" indicates the ***rotation*** in the vector.

Example 1: if $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

Sol:- Let $\bar{f} = xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$. Then

$$\text{curl } \bar{f} = \nabla \text{x } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$=\bar{i}\left(\frac{\partial}{\partial y}(-3yz^2)-\frac{\partial}{\partial z}(2x^2yz)\right)+\bar{j}\left(\frac{\partial}{\partial z}(xy^2)-\frac{\partial}{\partial x}(-3yz^2)\right)+\bar{k}\left(\frac{\partial}{\partial x}(2x^2yz)-\frac{\partial}{\partial y}(xy^2)\right)$$

$=\bar{i}\left(-3z^2-2x^2z\right)+\bar{j}(0-0)+\bar{k}(4xyz-2xy)$

= curl $\bar{f}$ = at (1,-1,1) = $-\bar{i}-2\bar{k}$.

Example 2: Find curl $\bar{f}$ where $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$. Then

grad ϕ= $\sum \bar{i}\frac{\partial \phi}{\partial x} = 3(x^2-yz)\bar{i}+3(y^2-zx)\bar{j}+3(z^2-xy)\bar{k}$

curl grad ϕ= ∇x grad ϕ= $3\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2-yz & y^2-zx & z^2-xy\end{vmatrix}$

$=3[\bar{i}(-x+x)-\bar{j}(-y+y)+\bar{k}(-z+z)]=\bar{0}$

curl $\bar{f}$ = $\bar{0}$.

Note: We can prove in general that curl (grad ϕ)=$\bar{0}$.(i.e) grad ϕ is always irrotational.

Example 3: Prove that curl $\bar{r}=\bar{0}$

Sol:- Let $\bar{r}$ = $x\bar{i}+y\bar{j}+z\bar{k}$

curl $\bar{r}=\sum \bar{i}x\frac{\partial}{\partial x}(\bar{r})=\sum(\bar{i}x\bar{i})=\bar{0}=\bar{0}$

Hence $\bar{r}$ is Irrotational vector.

Example 4: If f(r) is differentiable, show that curl { $\bar{r}$ f(r)} = $\bar{0}$ where $\bar{r}$ = $x\bar{i}+y\bar{j}+z\bar{k}$.

Sol: r = $\bar{r}=\sqrt{x^2+y^2+z^2}$ $r^2 = x^2+y^2+z^2$

$\Rightarrow 2r\frac{\partial r}{\partial x}=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}$, similarly $\frac{\partial r}{\partial y}=\frac{y}{r}, and \frac{\partial r}{\partial z}=\frac{z}{r}$

curl{$\bar{r}$ f(r)}= curl{f(r)($x\bar{i}+y\bar{j}+z\bar{k}$)}= curl $(x.f(r)\bar{i}+y.f(r)\bar{j}+z.f(r)\bar{k}$

$$=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xf(r) & yf(r) & zf(r)\end{vmatrix}=\sum \bar{i}\left[\frac{\partial}{\partial y}[zf(r)]-\frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum \bar{i}\left[zf^1(r)\frac{\partial r}{\partial y}-yf^1(r)\frac{\partial r}{\partial z}\right]=\sum \bar{i}\left[zf^1(r)\frac{y}{r}-yf^1(r)\frac{z}{r}\right]$$

= $\bar{0}$.

Example 5: Find constants a, b and c if the vector $\bar{f}$ = $(2x+3y+az)\bar{i}+(bx+2y+3z)\bar{j}+(2x+cy+3z)\bar{k}$ is Irrotational.

Sol:- Given $\bar{f}=(2x+3y+az)\bar{i}+(bx+2y+3z)\bar{j}+(2x+cy+3z)\bar{k}$

$$\text{Curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = (c-3)\bar{i}-(2-a)\bar{j}+(b-3)\bar{k}$$

If the vector is Irrotational then curl $\bar{f}$ = $\bar{0}$

$\Rightarrow$c-3 = 2-a=b-3 = 0

$\Rightarrow$c=3, a=2, b=3.

Scalar potential:-

If $\bar{f}$ is Irrotational, there will always exist a scalar function φ(x,y,z) such that $\bar{f}$ =gradϕ and the scalar function φ(x,y,z) is called ***scalar potential*** of $\bar{f}$.

It is easy to prove that, if $\bar{f}$ = gradϕ, then curl $\bar{f}$ = $\bar{0}$.

Hence ∇x $\bar{f}$ = $\bar{0}$ ⇔ there exists a scalar function ϕ such that $\bar{f}$ = ∇ϕ.

Note: This idea is useful when we study the "work done by a force" later.

Example 1: Show that the vector $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f}$ = $(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$

$$\text{Then curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix} = \sum i(-x+x) = \sum \bar{0} = \bar{0}$$

$\bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f}$ =∇ϕ.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z} = (x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x^2-yz \Rightarrow \phi = \int(x^2-yz)dx = \frac{x^3}{3}-xyz+f_1(y,z)......(1)$$

$$\frac{\partial\phi}{\partial y} = y^2-zx \Rightarrow \phi = \frac{y^3}{3}-xyz+f_2(z,x)......(2)$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x,y)......(3)$$

From (1), (2),(3), $\phi = \frac{x^3 + y^3 + z^3}{3} - xyz$

$$\therefore \quad \phi = \frac{1}{3}(x^3 + y^3 + z^3) - xyz + constant$$

Which is the required scalar potential.

Laplacian Operator ∇^2

We can see that $\nabla.\nabla\phi = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right) = \sum\frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$

Here the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called ***Laplacian operator***.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). If $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

Example 1: Prove that div(grad r^m)= $m(m+1)r^{m-2}$ (or) $\nabla^2(r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ then $r^2 = x^2+y^2+z^2$.

Differentiating w.r.t. 'x' partially, wet get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now grad(r^m) = $\sum \bar{i}\frac{\partial}{\partial x}(r^m) = \sum \bar{i}mr^{m-1}\frac{\partial r}{\partial x} = \sum \bar{i}mr^{m-1}\frac{x}{r} = \sum \bar{i}mr^{m-2}x$

div (grad r^m) = $\sum \bar{i}\frac{\partial}{\partial x}[mr^{m-2}x] = m\sum\left[(m-2)r^{m-3}\frac{\partial r}{\partial x}x + r^{m-2}\right]$

$= m\sum\left[(m-2)r^{m-4}x^2 + r^{m-2}\right] = m\left[(m-2)r^{m-4}\sum x^2 + \sum r^{m-2}\right]$

$= m[(m-2)r^{m-4}(r^2)+3r^{m-2}]$

$= m[(m-2)r^{m-2}+3r^{m-2}] = m[(m-2+3)r^{m-2}] = m(m+1)r^{m-2}$.

Hence $\nabla^2(r^m) = m(m+1)r^{m-2}$

Example 2: Show that $\nabla^2[f(r)] = \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f^{11}(r) + \frac{2}{r}f^1(r)$ where r = $|\bar{r}|$.

Sol: grad [f(r)] = $\nabla f(r) = \sum i\frac{\partial}{\partial x}[f(r)] = \sum i\, f^1(r)\frac{\partial r}{\partial x} = \sum i\, f^1(r)\frac{x}{r}$

$\nabla^2[f(r)]$ = div [grad $f(r)$]

$= \nabla.\nabla f(r)$

$= \sum \frac{\partial r}{\partial x}\left[f^1(r)\frac{x}{r} \right]$

$= \sum \frac{r\frac{\partial}{\partial x}[f^1(r)x] - f^1(r)x\frac{\partial}{\partial x}(r)}{r^2}$

$= \sum \frac{r\left(f^{11}(r)\frac{\partial r}{\partial x}x + f^1(r) \right) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$

$= \sum \frac{rf^{11}(r)\frac{x}{r}x + rf^1(r) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$

$= \frac{f^{11}(r)}{r^2}\sum x^2 + \frac{1}{r}f^1(r)\sum 1 - \frac{1}{r^3}f^1(r)\sum x^2$

$= \frac{f^{11}(r)}{r^2}r^2 + \frac{1}{r}f^1(r)(3) - \frac{1}{r^3}f^1(r)r^2$

$= f^{11}(r) + \frac{2}{r}f^1(r)$

Example 3: If ϕ satisfies Laplacian equation, show that $\nabla\phi$ is both solenoidal and Irrotational.

Sol: given $\nabla^2\phi = 0 \Rightarrow$div(grad ϕ)= 0 $\Rightarrow$ grad ϕ is solenoidal

We know that curl (grad ϕ) = $\bar{0} \Rightarrow$grad ϕ is always Irrotational.

Assignment-Cum-Tutorial Questions

SECTION-A

Objective Questions

1) Gradient of f(x,y,z) is
   (a) $\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$ (b) $\nabla f = \frac{\partial f}{\partial x}\bar{\imath} + \frac{\partial f}{\partial y}\bar{\jmath} + \frac{\partial f}{\partial z}\bar{k}$
   (b) (c) $\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ (d) None of these.
2) Divergence of is $\bar{F} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$ is
   (a) $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ (b) $\frac{\partial f_1}{\partial x}\bar{\imath} + \frac{\partial f_2}{\partial y}\bar{\jmath} + \frac{\partial f_3}{\partial z}\bar{k}$ (c) $\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}$ (d) $f_1 + f_2 + f_3$
3) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ then $\nabla.\bar{r} =$
   (a) 1 (b) 2 (c) 3 (d) 0
4) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ then $\nabla X\bar{r} =$
   (a) $\bar{\imath}$ (b) $\bar{\jmath}$ (c) $\bar{k}$ (d) $\bar{o}$
5) Given that $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ and $r = |\bar{r}|$, which of the following is false?

(a) $\nabla r^n = nr^{n-1}\bar{r}$ (b) $\nabla \sin(r) = cosr\frac{\bar{r}}{r}$ (c) $\nabla\frac{1}{r} = -\frac{\bar{r}}{r^3}$ (d) $\nabla logr = \frac{\bar{r}}{r}$

6) The Divergence of $\bar{F} = -sin\theta\bar{\imath} + cos\theta\bar{\jmath}$ is

(a) 0 (b) $-cos\theta - sin\theta$ (c) $\frac{sin2\theta}{r}$ (d) $-cos\theta\bar{\imath} - sin\theta\bar{\jmath}$

7) The Curl of $\bar{F} = x^3y^2\bar{\imath} - 3x^2y\bar{\jmath} + xyz\bar{k}$ is

(a) $xz\bar{\imath} - yz\bar{\jmath} + (-6xy + 2x^3y)\bar{k}$ (b) $3x^2y^2\bar{\imath} - 3x^2\bar{\jmath} + xy\bar{k}$

(c) $xz\bar{\imath} - yz\bar{\jmath} + (-6xy + 2x^3y)\bar{k}$ (d) $3x^2y^2\bar{\imath} - 3x^2\bar{\jmath} + xy\bar{k}$

8) If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find ϕ.

9) Find constants a,b,c so that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is Irrotational.

10) If $\bar{a}$ is constant vector then prove that grad $(\bar{a}.\bar{r}) = \bar{a}$

SECTION-B

Descriptive Questions

1) Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2, 3).
2) Find the directional derivative of $2xy+z^2$ at (1,-1,3) in the direction of $\bar{i} + 2\bar{j} + 3\bar{k}$.
3) Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.
4) Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at (2,1,-1).
5) Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$.
6) If ω is a constant vector, evaluate curl V where V = ω x $\bar{r}$.
7) If $f = (x^2+y^2+z^2)^{-n}$ then find div grad f and determine n if div grad $f = 0$.
8) Find the directional derivative of $\phi(x,y,z) = x^2yz+4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface $f(x,y,z) = x \log z - y^2$ at (-1,2,1).
9) Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).
10) If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change.
11) Evaluate the angle between the normals to the surface $xy = z^2$ at the points (4,1,2) and (3,3,-3).
12) Find the angle of intersection of the spheres $x^2+y^2+z^2 = 29$ and $x^2+y^2+z^2+4x-6y-8z-47 = 0$ at the point (4,-3,2).
13) Find the values of a and b so that the surfaces $ax^2-byz = (a+2)x$ and $4x^2y+z^3 = 4$ may intersect orthogonally at the point (1, -1,2).(or) Find the constants a and b so that surface $ax^2-byz=(a+2)x$ will orthogonal to $4x^2y+z^3=4$ at the point (1,-1,2).

SECTION-C

GATE/IES/Placement Tests/Other competitive examinations

1. If $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a}x\bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}.\bar{r})$.
2. Verify if ∇x$\left(\frac{\bar{A}X\bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}.\bar{A})\bar{r}}{r^{n+2}}$.
3. The magnitude of the gradinat of the function f = xyz^3 at (1,0,2) is
   a) 0 b) 3 c) 8 d) α
4. For the function $\phi = ax^2y - y^3$ to represent the velocity potential of an ideal fluid $\nabla^2\phi$ should be equal to zero J_n that case, the value of a has to be
   a) -1 b) 1 c) -3 d) 3
5. If $\bar{V} = 2xy\bar{i} + (2y^2 - x^2)\bar{j}$, the velocity vector, curl $\bar{v}$ will be
   a) $2y^2\bar{j}$ b) 6y$\bar{k}$ c) Zero d) $-4x\bar{k}$ **(GATE 1997)**
6. The maximum value of the directional derivative of the function $\nabla\phi = 2x^2 + 3y^2 + 5z^2$ at (1,1,-1) is **(GATE 2000)**
   a) 10 b) -4 c) $\sqrt{152}$ d) 152
7. The divergence of the vector field $(x-y)\bar{i} + (y-x)j + (x+y+z)\bar{k}$ is **(GATE 2008)**
   a) 0 b) 2 c) 1 d) 3

INTEGRAL TRANSFORMS AND MULTIPLE INTEGRALS

UNIT – VI

Vector Integration

Objectives:

- To Provide geometric and physical explanation of the integral of vector field over a curve.
- To apply the vector integral theorems to evaluate line, surface and volume integrals.

Syllabus:

Vector Integration:

Line, surface and volume integrals. Integral theorems: Greens - Stokes - Gauss Divergence Theorems (Without proof) and related problems. Applications: Work done, flux across the surface.

Sub Outcomes:

- Use line integrals to evaluate arc length, workdone by a vector field.
- Apply greens theorem to evaluate line integrals.
- Examine path dependence/independence of line integrals of vector field.
- Apply stokes and divergence theorems to evaluate surface and volume integrals.

Learning Material

Line Integral:

Any Integral which is evaluated along the curve is called Line Integral, and it is denoted by $\int_C \overline{F}.d\overline{r}$ where $\overline{F}$ is a vector point function, $\overline{r}$ is position vector and C is the curve.

Circulation:

If $\overline{v}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \overline{v}.d\overline{r}$ is called the circulation of $\overline{v}$ round the curve C.

Work done by a force:

- Work done by a force $\bar{F}$ during displacement from A to B is given by $\int_A^B \bar{F}.d\bar{r}$.

Q. If $\bar{F}(x,y,z) = x^3\bar{i} + y\bar{j} + z\bar{k}$ is the force field. Find the work done by $\bar{F}$ along the line from (1, 2, 3) to (3, 5, 7).

Solution The given line is $\frac{x-1}{3-1} = \frac{y-2}{5-2} = \frac{z-3}{7-3} = t$ (say)

$\therefore$ $x = 2t + 1,\ y = 3t + 2,\ z = 4t + 3$

Now, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = (2t+1)\hat{i} + (3t+2)\hat{j} + (4t+3)\hat{k}$

$\therefore$ $d\bar{r} = (2\hat{i} + 3\hat{j} + 4\hat{k})\,dt$

- At $A(1, 2, 3)$, $t = 0$ and at $(3, 5, 7)$, $t = 1$

$$\therefore \text{ Work done } = \int_C \bar{F}\cdot d\bar{r} = \int_0^1 [(2t+1)^3\,\hat{i} + (3t+2)\,\hat{j} + (4t+3)\hat{k}]$$

$$\cdot (2\hat{i} + 3\hat{j} + 4\hat{k})\,dt$$

$$= \int_0^1 [2(2t+1)^3 + 3(3t+2) + 4(4t+3)]\,dt$$

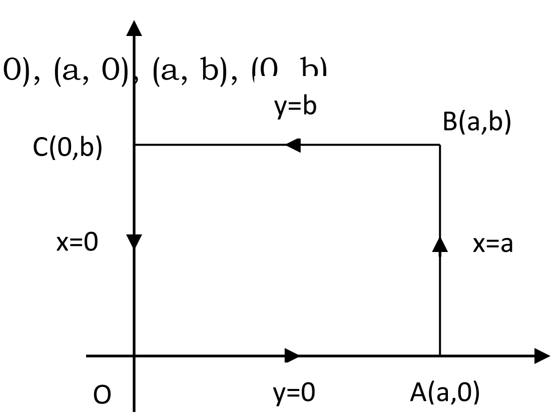
$$= \left[4t^4 + 8t^3 + \frac{37}{2}t^2 + 20t\right]_0^1$$

$$= 4 + 8 + \frac{37}{2} + 20 = \frac{101}{2}$$

- **Q.** Evaluate $\oint_C (x^2 + y^2)dx - 2xydy$ where C is a rectangle with vertices (0, 0), (a, 0), (a, b), (0, b).

Solution: Draw the given rectangle with vertices (0, 0), (a, 0), (a, b), (0, b)

Given integral $\oint_C (x^2 + y^2)dx - 2xydy$



$$\oint_C (x^2+y^2)dx - 2xydy = \int_{OA} (x^2+y^2)dx - 2xydy \int_{AB} (x^2+y^2)dx - 2xydy + \int_{BC} (x^2+y^2)dx - 2xydy + \int_{CO} (x^2+y^2)dx - 2xydy$$

Along OA: $y = 0 \Rightarrow dy = 0$

x varies from 0 to a

$$\int_{OA} (x^2+y^2)dx - 2xydy = \int_0^a x^2 dx = \frac{a^3}{3}$$

Along AB: $x = a \Rightarrow dx = 0$

y varies from 0 to b

$$\int_{AB} (x^2+y^2)dx - 2xydy = \int_0^b -2ady = -ab^2$$

Along BC: $y = b \Rightarrow dy = 0$

x varies from a to 0

$$\int_{BC} (x^2+y^2)dx - 2xydy = \int_a^0 (x^2+b^2)dx = -\frac{a^3}{3} - ab^2$$

Along CO: $x = 0 \Rightarrow dx = 0$

y varies from b to 0

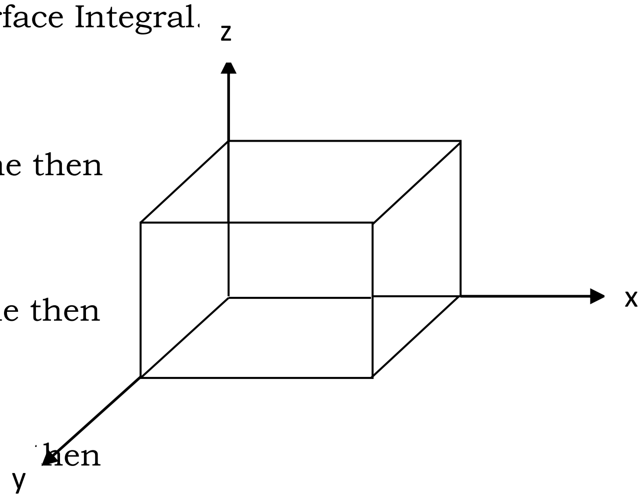
$$\int_{CO} (x^2+y^2)dx - 2xydy = \int_b^0 0dy = 0$$

$$\therefore \oint_C (x^2+y^2)dx - 2xydy = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0$$

$$\therefore \oint_C (x^2+y^2)dx - 2xydy = -2ab^2$$

Surface Integral :

- The Integral which is evaluated over a surface is called Surface Integral.
- If S is any surface and $\bar{n}$ is the outward drawn unit normal vector to the surface S then $\int_S \bar{F}.\bar{n}ds$ is called the Surface Integral.



Note:

- Let R_1 be the projection of S on xy-plane then
  $$\int_S \bar{F}.\bar{n}ds = \iint_{R_1} \frac{\bar{F}.\bar{n}dxdy}{|\bar{n}.\bar{k}|}$$
- Let R_2 be the projection of S on yz-plane then
  $$\int_S \bar{F}.\bar{n}ds = \iint_{R_2} \frac{\bar{F}.\bar{n}dydz}{|\bar{n}.\bar{i}|}$$
- Let R_3 be the projection of S on zx-plane then

$$\int_S \overline{F}.\overline{n}ds = \iint_{R_3} \frac{\overline{F}.\overline{n}dzdx}{|\overline{n}.\overline{j}|}$$

Q. Evaluate $\iint_S (yz\overline{i} + zx\overline{j} + xy\overline{k}).d\overline{S}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

Solution We have $\phi = x^2 + y^2 + z^2 - a^2$. Then,

$$\nabla\phi = \hat{i}\cdot 2x + \hat{j}\cdot 2y + \hat{k}\cdot 2z \text{ and } |\nabla\phi| = 2a$$

$\therefore$ The unit normal vector to the surface ϕ is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

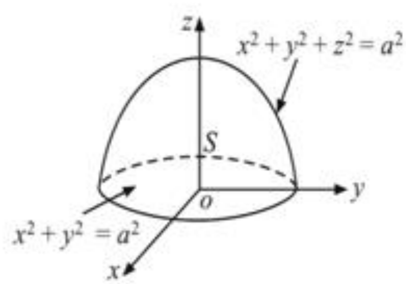
$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2a} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

Also, $$\bar{F}\cdot\hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right) = \frac{xyz + xyz + xyz}{a}$$

$$= \frac{3xyz}{a} \qquad \text{(i)}$$

$\therefore$ $$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R \bar{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \qquad \text{(ii)}$$

where R is the projection of S on xy-plane. This projection R is bounded by the x-axis, y-axis and the circle $x^2 + y^2 = a^2$, $z = 0$. Hence, x varies from 0 to a and y from 0 to $\sqrt{a^2 - x^2}$ (Figure 5.8).



Therefore, $|\hat{n}\cdot\hat{k}| = \left|\dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\cdot\hat{k}\right|$

$$= \frac{z}{a}$$

Hence, Eq. (ii) becomes

$$\iint_S \bar{F}\cdot\hat{n}dS = \iint_R \frac{3xyz}{a}\,\frac{dx\,dy}{(z/a)} = 3\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

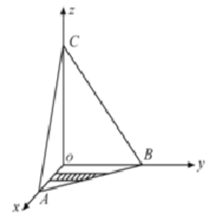
$$= 3\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \frac{3}{2}\int_0^a x(a^2 - x^2)\,dx$$

$$= \frac{3}{2}\left[\frac{x^2a^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{3a^4}{8}$$

Q. Evaluate $\iint_S \bar{F}.\bar{n}ds$, where $\bar{F} = 6z\bar{i} - 4\bar{j} + y\bar{k}$ and S is the portion of the plane $2x+3y+6z=12$, which is in the first octant.

Solution The surface is $\phi = 2x + 3y + 6z - 12$.

$$\therefore \quad \nabla\phi = \hat{i}\cdot 2 + \hat{j}\cdot 3 + \hat{k}\cdot 6$$



and $$|\nabla\phi| = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$\therefore \quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

$$\bar{F}\cdot\hat{n} = (6z\hat{i} - 4\hat{j} + y\hat{k})\cdot\frac{(2\hat{i}+3\hat{j}+6\hat{k})}{7} = \frac{12z - 12 + 6y}{7}$$

$$= \frac{1}{7}\left[12\,\frac{(12-2x-3y)}{6} - 12 + 6y\right] = \frac{1}{7}[12 - 4x - 6y]$$

$$= \frac{1}{7}(12 - 4x)$$

Finally, $$\iint_S \bar{F}\cdot\hat{n}\,dS = \iint_R \bar{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

where R is the projection of S on the xy-plane. Hence, R is bounded by x-axis, y-axis and $z = 0$. Now,

$$|\hat{n}\cdot\hat{k}| = \left|\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\cdot\hat{k}\right| = \frac{6}{7}$$

Therefore, $$\iint_S \bar{F}\cdot\hat{n}dS = \iint_R \frac{1}{7}(12-4x)\frac{dx\,dy}{(6/7)} = \iint_R \frac{2}{3}(3-x)\,dx\,dy$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} \frac{2}{3}(3-x)\,dy\,dx$$

$$= \int_0^6 \frac{2}{3}(3-x)\,[y]_0^{\frac{12-2x}{3}}\,dx$$

$$= \frac{2}{3}\int_0^6 (3-x)\left(\frac{12-2x}{3}\right)dx$$

$$= \frac{4}{9}\int_0^6 (18 - 6x - 3x + x^2)\,dx$$

$$= \frac{4}{9}\left[18x - \frac{9x^2}{2} + \frac{x^3}{3}\right]_0^6$$

$$= \frac{4}{9}\left[18(6) - \frac{9\times 6\times 6}{2} + \frac{(6)^3}{3}\right] = \frac{4}{9}[108 - 162 + 72]$$

$$= \frac{4}{9}[18] = 8$$

Volume Integral :

The Integral which is evaluated over a volume is called Volume Integral.

If $\overline{F}$ is a vector point function bounded by the region R with volume V, then $\int_V \overline{F}dv$ is called as Volume Integral.

Q. If $\overline{F} = 2z\overline{i} - x\overline{j} + y\overline{k}$ **,** evaluate $\iiint_V \overline{F}dV$ where V is the volume bounded by the surfaces $x = 0,\quad y = 0,\quad x = 2,\quad y = 4,\ \ z = x^2,\ z = 2$

Solution. $\iiint_V \overline{F}\,dv = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$

$$= \int_0^2 dx\int_0^4 dy\int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\,dz \quad = \int_0^2 dx\int_0^4 dy\,[z^2\hat{i} - xz\hat{j} + yz\hat{k}]_{x^2}^2$$

$$= \int_0^2 dx\int_0^4 dy\,[4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k}]$$

$$= \int_0^2 dx\left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4y\hat{i} + x^3y\hat{j} - \frac{x^2y^2}{2}\hat{k}\right]_0^4$$

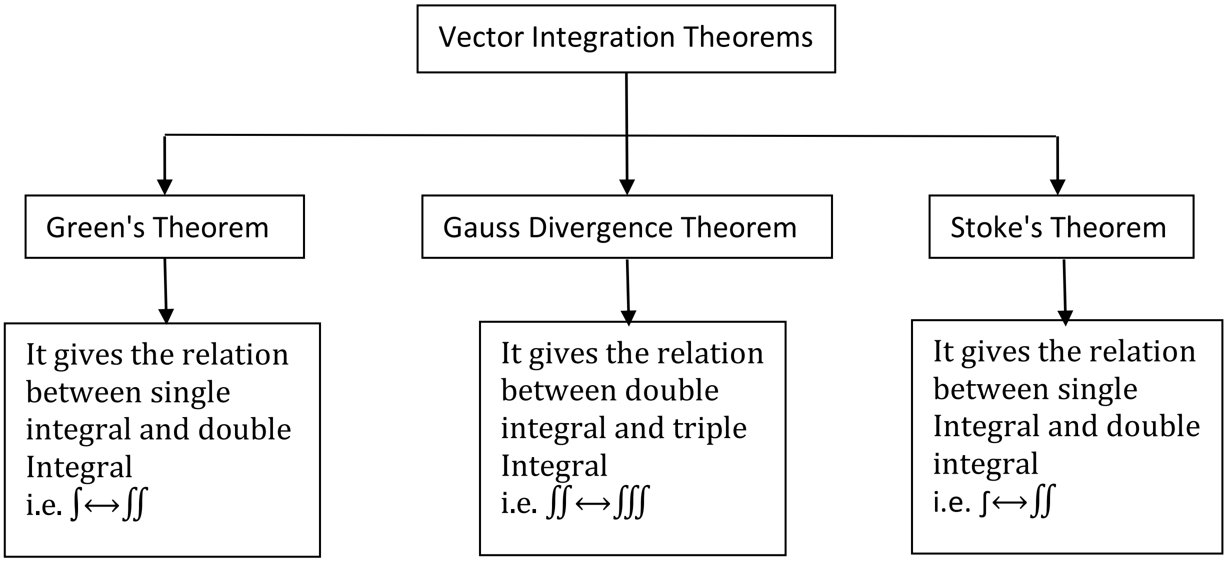
$$= \int_0^2 (16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k})\,dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k}$$

$$= \frac{32\hat{i}}{5} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$

Vector Integration Theorems



Why these theorems are used?

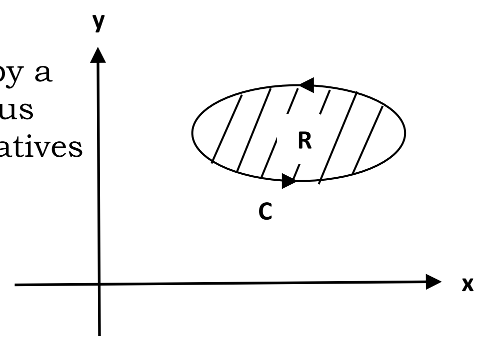
While evaluating Integration (single/double/triple) problems, we come across some Integration problems where evaluating single integration is too hard, but if we change the same problem in to double integration, the Integration problem becomes simple.

In such cases,

- We use Greens Theorem (if the given surface is xy-plane) (or) Stokes Theorem (for any plane).
- If we want to change double integration problem in to triple integral, we use Gauss Divergence Theorem.

Greens Theorem:

If R is a closed region in xy-plane bounded by a simple closed curve C and If M and N are continuous functions of x and y , and having continuous derivatives in R, then



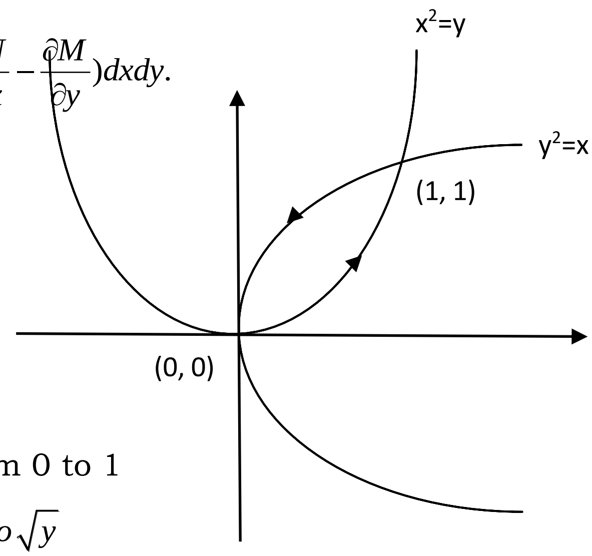
$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy.$$

Where C is traversed in the positive direction (i.e. anti clock-wise).

Note: Greens Theorem is used if the given surface is in xy -plane only.

Q. Using Green's theorem evaluate $\int_C (2xy - x^2)dx + (x^2 + y^2)dy$ where C is the closed curve of the region bounded by y= x^2 and y^2 = x.

***Solution*: Greens Theorem:** $\oint_C M dx + N dy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dxdy.$



Step-1: Draw the region bounded by y= x² and y² = x.

Intersecting points are (0, 0) , (1, 1)

Step-2: Identify the limits x varies from 0 to 1

y varies from $x^2 \; to \sqrt{x}$

(Or) We can take the limits y varies from 0 to 1

x varies from $y^2 \; to \sqrt{y}$

Step-3: By comparing given integral with LHS of Greens theorem, Identify

$$M = 2xy - x^2 \qquad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

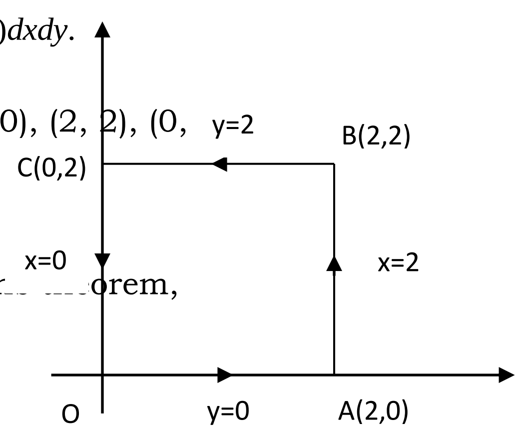
By Greens theorem $\oint_C M dx + N dy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dxdy.$

$$\int_C (2xy - x^2) dx + (x^2 + y^2) dy = \int_0^1 \int_{x^2}^{\sqrt{x}} 0 \, dxdy$$

$$= 0$$

Q. Verify Greens theorem in the plane for $\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy$ where C is a square with vertices (0, 0), (2, 0), (2, 2), (0, 2).

***Solution*: Greens Theorem:** $\oint_C M dx + N dy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dxdy.$



Step-1: Draw the given square with vertices (0, 0), (2, 0), (2, 2), (0,

Given integral $\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy$

Step-2: By comparing given integral with LHS of Greer... ...orem, Identify

$$M = x^2 - xy^3 \qquad N = y^2 - 2xy$$

$$\frac{\partial M}{\partial y} = -3xy^2 \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3xy^2 - 2y$$

Evaluation of LHS: $\oint_C Mdx + Ndy$ **:**

$$\oint_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy + \int_{CO} Mdx + Ndy$$

Along OA: $y = 0 \Rightarrow dy = 0$

x varies from 0 to 2

$$\int_{OA} Mdx + Ndy = \int_{OA} (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$= \int_0^2 x^2 dx$$

$$= \frac{8}{3}$$

Along AB: $x = 2 \Rightarrow dx = 0$

y varies from 0 to 2

$$\int_{AB} Mdx + Ndy = \int_{AB} (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$= \int_0^2 (y^2 - 4y)dy$$

$$= -\frac{16}{3}$$

Along BC: $y = 2 \Rightarrow dy = 0$

x varies from 2 to 0

$$\int_{BC} Mdx + Ndy = \int_{BC} (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$= \int_2^0 (x^2 - 8x)dx$$

$$= \frac{40}{3}$$

Along CO: $x = 0 \Rightarrow dx = 0$

y varies from 2 to 0

$$\int_{CO} Mdx + Ndy = \int_{CO} (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$= \int_2^0 y^2 dy$$

$$= -\frac{8}{3}$$

$$\therefore \oint_C Mdx + Ndy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3}$$

$$\oint_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy = 8$$

Evaluation of RHS: $\iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy$ **:**

From Region x varies from 0 to 2

y varies from 0 to 2

$$\iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy = \int_0^2 \int_0^2 (3xy^2 - 2y)dxdy$$

$$= \int_0^2 (3x\frac{y^3}{3} - 2\frac{y^2}{2})_0^2 dx$$

$$= \int_0^2 (8x - 4)dx$$

$$= 8$$

$$\therefore \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy = 8$$

$$\therefore \oint_C Mdx + Ndy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy.$$

Hence Greens theorem verified.

Gauss Divergence Theorem:

Let S be a closed surface enclosing a volume V. If $\overline{F}$ is a continuously differentiable vector point function, then

$$\int_V div\overline{F}dv = \int_S \overline{F}.\overline{n}ds$$

Where $\bar{n}$ is the outward drawn normal vector at any point of S.

Note: Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

Then $div\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$dv = dxdydz$

$$\iiint_V div\bar{F}dv = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

Q. Use Gauss' theorem to evaluate the surface integral $\iint_S \bar{F}.\bar{n}ds$ where F is the vector field $x^2yi + 2xyj + z^3k$ and S is the surface of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

***Solution*: Gauss Divergence Theorem:** $\int_V div\bar{F}dv = \int_S \bar{F}.\bar{n}ds$

Given vector field is

$$\bar{F} = x^2yi + 2xyj + z^3k$$

Thus $F_1 = x^2y$ $\quad F_2 = 2xy$ $\quad F_3 = z^3$

$$div\bar{F} = \nabla.\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2xy + 2x + 3z^2$$

Here the limits are x varies from 0 to 1

y varies from 0 to 1

z varies from 0 to 1

By Gauss Divergence Theorem $\int_S \bar{F}.\bar{n}ds = \int_V div\bar{F}dv$

$$= \int_0^1\int_0^1\int_0^1 (2xy + 2x + 3z^2)dxdydz$$

$$= \int_0^1\int_0^1 (y + 1 + 3z^2)dydz$$

$$= \int_0^1 \left(\frac{3}{2} + 3z^2\right) dz$$

$$= \frac{11}{6}$$

Stokes Theorem:

Let S be an open surface bounded by a closed curve C. If $\overline{F}$ is any continuously differentiable vector point function then

$$\int_C \overline{F}.d\overline{r} = \iint_S Curl\,\overline{F}.\overline{n}ds$$

Where C is traversed in the positive direction and $\overline{n}$ is the outward drawn unit normal vector at any point of the surface S.

Note: Stokes Theorem is used for any surface (or) any plane (xy-plane, yz-plane, zx -plane)

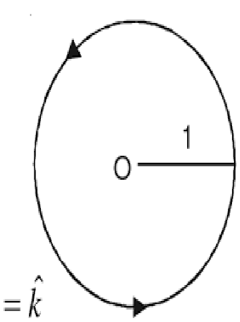
Q. Using Stoke's theorem or otherwise, evaluate $\int_C [(2x-y)dx - yz^2dy - y^2zdz]$, where C is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.

Solution. $\int_c [(2x-y)\,dx - yz^2dy - y^2z\,dz]$

$$= \int_c [(2x-y)\,\hat{i} - yz^2\,\hat{j} - y^2z\,\hat{k}]\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

By Stoke's theorem $\oint \overline{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\overline{F}\cdot\overline{n}\,ds$ ...(1)

$$\text{Curl}\,\overline{F} = \nabla\times\overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$



$$= (-2\,yz + 2\,yz)\,\hat{i} - (0-0)\,\hat{j} + (0+1)\,\hat{k} = \hat{k}$$

Putting the value of curl $\overline{F}$ in (1), we get

$$= \iint \hat{k}\cdot\hat{n}\,ds = \iint \hat{k}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \iint dx\,dy = \text{Area of the circle} = \pi \qquad \left[\because ds = \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}\right]$$

Q. Verify Stokes theorem for the vector field $\overline{F} = (2x - y)\overline{i} - yz^2\overline{j} - y^2z\overline{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane.

***Solution*: Stokes theorem:** $\int_C \overline{F}.d\overline{r} = \iint_S Curl\,\overline{F}.\overline{n}ds$

Evaluation of $\int_C \overline{F}.d\overline{r}$ **:**

On C, $x^2 + y^2 = 1$

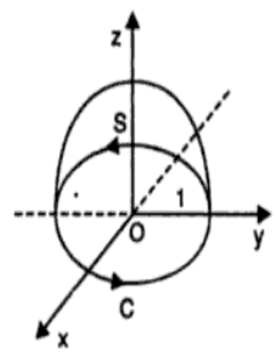
i.e., $x = \cos t,\ y = \sin t$ and $z = 0$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$\vec{F}$ on $C = (2x - y)\vec{i}$

$$\therefore \quad \vec{F}\cdot d\vec{r} = (2x - y)\,dx$$



$$\text{L.H.S.} = \int_C \vec{F}\cdot d\vec{r} = \int_C (2x - y)\,dx$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \int_0^{2\pi} (\sin^2 t - \sin 2t)\,dt$$

$$= 4\int_0^{\pi/2} \sin^2 t\,dt - \int_0^{2\pi} \sin 2t\,dt = 4\cdot\frac{1}{2}\cdot\frac{\pi}{2} - 0 = \pi.$$

Evaluation of $\iint_S Curl\,\overline{F}.\overline{n}ds$ **:**

Now, $\quad \text{curl}\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \vec{k}$

$$\text{curl}\ \vec{F} \cdot \vec{n} = \vec{k} \cdot \vec{n} = \vec{n} \cdot \vec{k}$$

$\therefore \quad \text{R.H.S.} = \int_S \text{curl}\ \vec{F} \cdot \vec{n}\, dS = \int_S \vec{n} \cdot \vec{k}\, dS$

$$= \iint_{S_1} \vec{n} \cdot \vec{k} \frac{dx\, dy}{\vec{n} \cdot \vec{k}},$$ where S_1 is the projection of S on xy-plane.

$$= \iint_{S_1} dx\, dy = \int_0^{2\pi} \int_0^1 r\, dr\, d\theta$$ (taking in Polar coordinates)

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{1}{2} \cdot \int_0^{2\pi} d\theta = \frac{1}{2} \cdot 2\pi = \pi$$

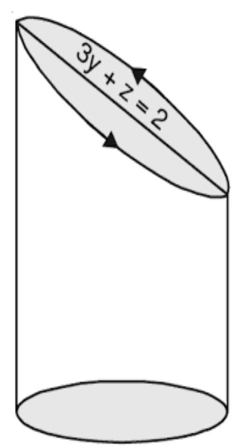
$\therefore \quad$ **L.H.S. = R.H.S.**

Hence Stoke's theorem is verified.

Q. Evaluate $\int_C \bar{F}.d\bar{r}$,where $\bar{F}(x, y, z) = -y^2\bar{i} + x\bar{j} + z^2\bar{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Solution. $\oint_C \vec{F} . \vec{dr} = \iint_S \text{curl}\ \bar{F} . \hat{n}\, ds = \iint_S \text{curl}\ (-y^2\hat{i} + x\hat{j} + z^2\hat{k})\, \hat{n}\, ds \quad ...(1)$

$F(x, y, z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ *(By Stoke's Theorem)*



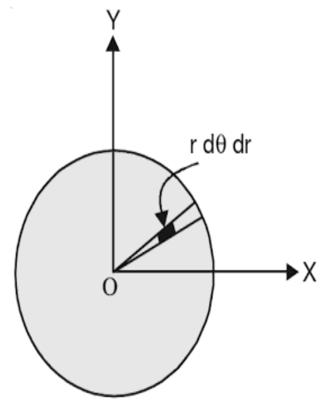
$$\text{Curl}\ \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1+2y) = (1+2y)\hat{k}$$

Normal vector $= \nabla \bar{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y + z - 2) = \hat{j} + \hat{k}$$

Unit normal vector $\hat{n} = \frac{\hat{j}+\hat{k}}{\sqrt{2}}$



$$ds = \frac{dx\,dy}{\hat{\eta}\,.\,\hat{k}}$$

On putting the values of curl $\vec{F}, \hat{n}$ and ds in (1), we get

$$\int_C \vec{F}\,.\,\vec{dr} = \iint_S (1+2y)\,\hat{k}\,.\,\frac{\hat{j}+\hat{k}}{\sqrt{2}}\;\frac{dx\,dy}{\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right).\hat{k}}$$

$$= \iint \frac{1+2y}{\sqrt{2}}\;\frac{dx\,dy}{\frac{1}{\sqrt{2}}} = \iint (1+2y)\,dx\,dy = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^1 (r+2r^2\sin\theta)\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta\left[\frac{r^2}{2}+\frac{2r^3}{3}\sin\theta\right]_0^1 = \int_0^{2\pi}\left[\frac{1}{2}+\frac{2}{3}\sin\theta\right]d\theta$$

$$= \left[\frac{\theta}{2}-\frac{2}{3}\cos\theta\right]_0^{2\pi} = \left(\pi-\frac{2}{3}-0+\frac{2}{3}\right) = \pi$$

Q. Verify Stokes theorem for the vector field $\bar{F} = (x^2-y^2)\bar{i} + 2xy\bar{j}$ integrated round the rectangle in the plane z=0 and bounded by the lines *x=0, y=0, x=a, y=b.*

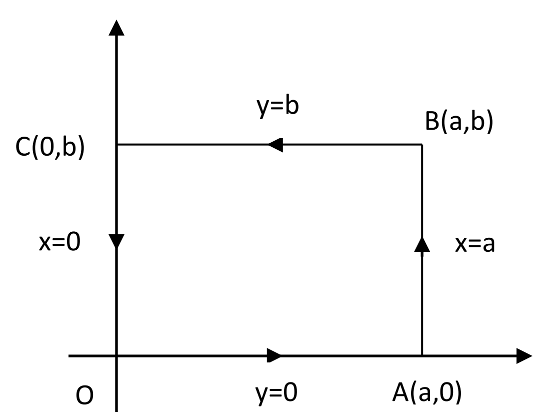
***Solution*: Stokes theorem:** $\int_C \bar{F}.d\bar{r} = \iint_S Curl\,\bar{F}.\bar{n}ds$

Evaluation of LHS= $\int_C \bar{F}.d\bar{r}$ **:**

Draw the given rectangle in the plane z=0 and bounded by the lines *x=0, y=0, x=a, y=b.*

i.e rectangle with vertices (0, 0), (a, 0), (a, b), (0, b).

$$\oint_C \bar{F}.d\bar{r} = \int_{OA} \bar{F}.d\bar{r} + \int_{AB} \bar{F}.d\bar{r} + \int_{BC} \bar{F}.d\bar{r} + \int_{CO} \bar{F}.d\bar{r}$$



We know that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Thus $\bar{F}.d\bar{r} = (x^2-y^2)dx + 2xydy$

Along OA: $y = 0 \Rightarrow dy = 0$

x varies from 0 to a

$$\int_{OA} \bar{F}.d\bar{r} = \int_{OA} (x^2 - y^2)dx + 2xydy = \int_0^a x^2 dx = \frac{a^3}{3}$$

Along AB: $x = a \Rightarrow dx = 0$

y varies from 0 to b

$$\int_{AB} \bar{F}.d\bar{r} = \int_{AB} (x^2 - y^2)dx + 2xydy = \int_0^b 2ady = ab^2$$

Along BC: $y = b \Rightarrow dy = 0$

x varies from a to 0

$$\int_{BC} \bar{F}.d\bar{r} = \int_{BC} (x^2 - y^2)dx + 2xydy = \int_a^0 (x^2 - b^2)dx = -\frac{a^3}{3} + ab^2$$

Along CO: $x = 0 \Rightarrow dx = 0$

y varies from b to 0

$$\int_{CO} \bar{F}.d\bar{r} = \int_{CO} (x^2 - y^2)dx + 2xydy = \int_b^0 0dy = 0$$

$$\therefore \oint_C \bar{F}.d\bar{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$\therefore \oint_C \bar{F}.d\bar{r} = 2ab^2$$

Evaluation of RHS= $\iint_S Curl\bar{F}.\bar{n}ds$

Now, $\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\,\vec{k}$

For the surface, $S, \vec{n} = \vec{k}$

$\therefore \quad \text{curl } \vec{F} \cdot \vec{n} = 4y$

$$\text{R.H.S.} = \iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS = \int_0^a \int_0^b 4y\, dy\, dx$$

$$= \int_0^a 4 \cdot \left[\frac{y^2}{2}\right]_0^b dx = 2b^2 \int_0^a dx = 2ab^2$$

$\therefore$ L.H.S. = R.H.S. Hence the Stoke's theorem is verified

Assignment-Cum-Tutorial Questions

SECTION-A

I) *Objective Questions*

1. $Evaluate \int_C \bar{F}.d\bar{r}$,where $\bar{F} = (x+y)\bar{i} + (y-x)\bar{j}$ and C is
   (i) the parabola $y^2=x$ between the points (1, 1) and (4, 2).
   (ii) the straight line joing the points (1, 1) and (4, 2).
2. The value of $\int_C \bar{F}.d\bar{r}$, where $\bar{F} = i + j + k$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $x = t, y = t, z = t, \quad 0 \le t \le 1$ is the curve C.
3. Use Green's Theorem to evaluate $\int_C y^2dx + xydy$ for C: boundary of the region lying between the graphs of $y = 0, y = \sqrt{x}$ *and* $x = 9$.
   [ ]
   a) $-\frac{81}{2}$ b) $\frac{81}{4}$ c) $\frac{243}{4}$ d) $-\frac{81}{4}$
4. Use Stokes's Theorem to evaluate $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = y^2\bar{i} + z^2\bar{j} + 2xy\bar{k}$ around the triangle with vertices (2, 0, 0), (0, 1, 0), and (0, 0, 4) . [ ]
   a)-2 b)-6 c)2 d)6
5. If E is the solid region bounded by the planes x=0,y=0,z=0 and $2x + 2y + z = 4$ then the triple integral $\iiint_E ydV$ is
   [ ]
   a) $-\frac{1}{3}$ b)4 c) $\frac{4}{3}$ d) $\frac{2}{3}$
6. $\int_s \bar{r} \cdot \bar{n} \, ds$ = [ ]
   a) V b) 2V c) 3V d) $\frac{3}{2}$V
7. $\int_s (\bar{r} \times \bar{n}) \, ds$ = [ ]
   a) 0 b) r c) 1 d) -1
8. If S is any closed surface enclosing a volume V and $\bar{F} = x\,\bar{i} + 2y\,\bar{j} + 3z\,\bar{k}$ then $\int\int_s \bar{F} \cdot \bar{n} \, ds$ = [ ]
   a) V b) 3V c) 6V d) 2V
9. From green's theorem $\int Pdx + Qdy$ = [ ]
   a) $\iint\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right)dx\,dy$ b) $\iint\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dx\,dy$ c) $\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$ d) 0

10. $\int_c (ax\bar{i} + by\bar{j} + cz\bar{k}) \cdot \bar{n}\ ds$ = [ ]

a) $\frac{2\pi}{3}(a+b+c)$ b) $\frac{4\pi}{3}(a+b+c)$ c) $(a+b+c)$ d) $\frac{3\pi}{4}(a+b+c)$

11. The work done by a force $\bar{F} = (3x^2 + 6y)\ \bar{i} - 14yz\ \bar{j} + 20xz^2\ \bar{k}$ along the lines from (0, 0, 0) to (1, 0, 0) is [ ]

a) 1/2 b) 3/2 c) 0 d) 1

12. For any closed surface S, $\iint_s \text{curl}\ \bar{F}\ \bar{n}\ ds$ = [ ]

a) 0 b) 2F c) $\bar{n}$ d) $\int \bar{F} \cdot d\bar{r}$

SECTION-B

II) Descriptive Questions

1. If $\bar{F} = xy\,\bar{i} - z\,\bar{j} + x^2\bar{k}$ and C is the Curve $x = t^2,\ y = 2t,\ z = t^3$ from $t = 0$ to $t = 1$. Evaluate $\int_C \bar{F}.d\bar{r}$.
2. Compute the line integral $\int (y^2dx - x^2dy)$ round the triangle whose vertices are (1,0), (0,1) and (-1,0) in the xy-plane.
3. Evaluate $\int_C \bar{f}.d\bar{r}$ where $\bar{f} = x^2 i + y^2 j$ and curve c is the arc of the parabola $y=x^2$ in the xy-plane from (0,0) to (1,1).
4. Show that $\bar{F} = (2xy + z^3)i + x^2 j + 3xz^2 k$ is a conservative force field. Find the scalar potential and the work done in moving an object in this field from (1, -2, 1) to (3, 1, 4).
5. Verify divergence theorem for **F** = 4xz ***i*** – y^2 **j** + yz **k** taken over the cube bounded by x = 0, x = 1; y = 0 , y = 1; z = 0 , z = 1.
6. Verify Green's theorem in the xy - plane for $\oint_C [(xy^2 - 2xy)dx + (x^2y + 3)dy]$ around the boundary C of the region enclosed by $y^2 = 8x$, x = 2 and the x-axis.
7. Verify Green's theorem for $\oint_C [(xy + y^2)dx + x^2dy]$. where C is a bounded by y=x and $y=x^2$.

8. Verify Stokes theorem for F = (y-z+2)i+(yz+4)j-xzk where S is the surface of the cube x=0, y=0,z=0,x=2,y=2,z=2 above the xy-plane.
9. Evaluate by Green's theorem $\oint_C [(y-\sin x)dx + \cos x dy]$ where 'C' is the triangle enclosed by the lines y = 0, x = π/2 and πy = 2x.
10. Apply Gauss divergence theorem to evaluate $\iint_S \overline{F}.\overline{n}ds$ where $F = yi + xj + zk$ and S is the surface of the cylindrical region bounded by $x^2 + y^2 = 9$ *and* $z = 0$ *and* $z = 2$.
11. Use Gauss divergence theorem to evaluate $\iint_S (yz^2 i + zx^2 j + 2z^2 k).Nds$ where S is the surface bounded by the xy-plane and the upper half of the sphere $x^2+y^2+z^2=a^2$ above this plane.
12. Evaluate the integral $I = \iint_S x^3 dydz + x^2 ydzdx + x^2 zdxdy$ using divergence theorem, where S is the surface consisting of the cylinder $x^2 + y^2 = a^2 (0 \le z \le b)$ and the circular disks z=0 and $z = b(x^2 + y^2 \le a^2)$.
13. Apply Stoke's Theorem to evaluate $\oint_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ when C is the boundary of the triangle with vertices (2,0,0), (0,3,0) and (0,0,6).
14. Evaluate by Stokes theorem $\oint_C [(x+y)dx + (2x-z)dy + (y+z)dz]$, where C is the boundary of the triangle vertices (0,0,0), (1,0,0) and (1,1,0).
15. If $f = (x^2 + y - 4)i + 3xyj + (2xz + z^2)k$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = 16$. Show by using Stokes theorem that $\int_S curl\ f.nds = 2\pi a^3$

Section-C

GATE/IES/Placement Tests/Other competitive examinations

1. If S is the surface of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and
   $ax + by + cz = 1$. Show that $\int_S r.nds = \frac{1}{2abc}$.
2. If ϕ is a scalar point function, using Stoke's theorem prove that $Curl(grad\phi) = 0$.
3. Consider points P and Q in the xy plane with P=(1,0) and Q = (0,1) The line integral $\int_P^Q xdx + ydy$ along the semicircle with line segment PQ as its diameter **(GATE 2010)**
   a) -1 b) 0 c) 1
   d) depends on direction C clock wise or anti clock wise) of the semi circle

4. A triangle ABC consists of vertex points A(0,0) B(1,0) and C(0,1). The value of the integral $\iint 2x dx dy$ over the triangle is

(GATE 1997)

a) 1 b) $\frac{1}{3}$ c) $\frac{1}{8}$ d) $\frac{1}{9}$

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