UNIT –I

Objective:

To introduce the students to the concepts of mechanics, resultant of various force systems and equilibrium of systems of forces.

Syllabus:

Introduction: Units- Scalar and Vector quantities.

Types of forces: Coplanar, concurrent, parallel and collinear forces – principle of transmissibility.

Coplanar concurrent forces: Parallelogram Law of force system (Cosine rule), Triangular Law of three force system, Lami's theorem (Sine rule) – Polygon law of forces – Composition &Resolution of forces – Method of projections, Resultant of concurrent force systems – Simple examples.

Equilibrium of system of forces: Support conditions – Roller, Hinge and fixed supports – Mode of reactions.

Free Body Diagram – Equilibrium – Equilibrium of coplanar force systems and condition of equilibrium.

Moment of a force: Varignon theorem – Moment of a force in plane and its application – Simple problems.

Parallel forces: Like and unlike parallel force systems, Couple and its action in plane, Beams subjected to loads – End reactions.

Space force systems: Concurrent space force systems – Cartesian coordinates, Components of space forces on to x, y and z axes using method of projections, Resultant and its position.

Learning Outcomes:

Student will be able to

- Distinguish Scalar and Vector quantities.
- State the principle of transmissibility
- Define various types of forces
- Prove Parallelogram law of two force system
- State Lami's Theorem
- Determine the magnitude & direction of the resultant of concurrent force systems.
- Draw the Free Body diagrams.
- Explain the conditions of equilibrium.
- Calculate Moment of a force in a plane.
- Determine the resultant of a System of Concurrent Space force systems.

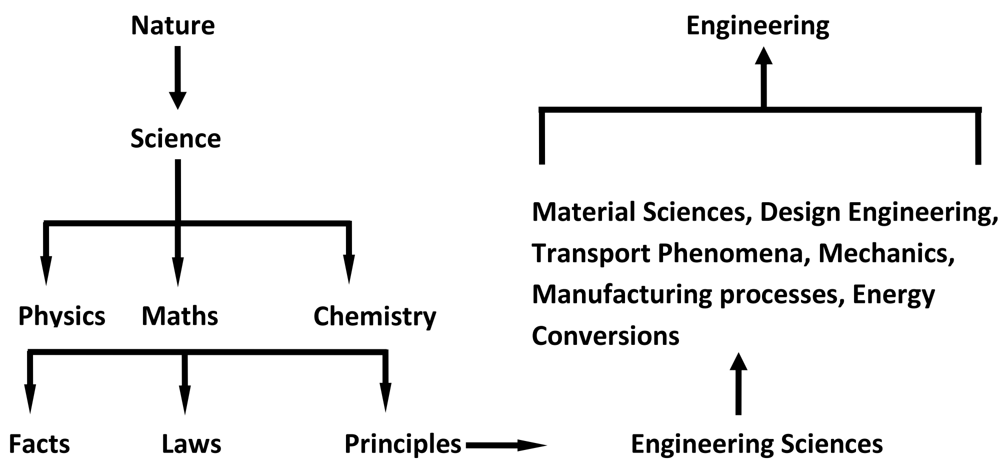
Learning Material

UNIT 1

Engineering is an activity concerned with the creation of new systems for benefit of mankind.

Creation of new systems is thus basic to all engineering disciplines.

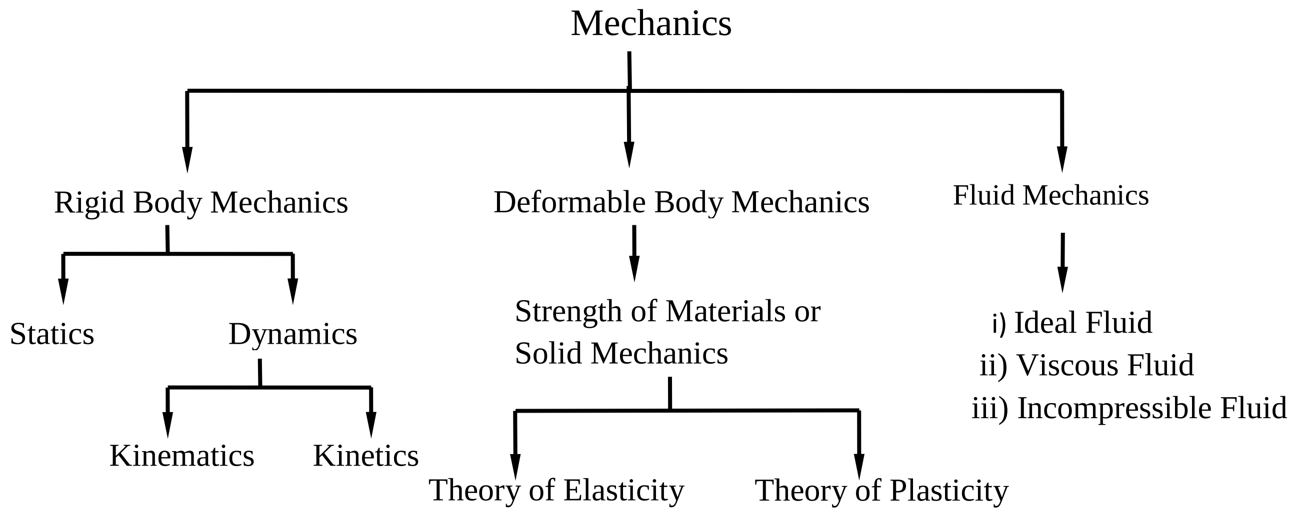
Engineering is also application of science. Science is concerned with a systematic understanding and gathering the facts, laws and principles governing natural phenomena.



<u>History of classical mechanics</u>:

Newton	(1642-1727)	→ Newtonian →	Non-relativistic
Lagrange	(1736-1813)	→	Relative Mechanics
Hamilton	(1805-1865)		↓
D’Alembert	(1717-1783)		Lorentz (1853-1928)
Kepler	(1517-1630)		Einstein (1878-1955)
Archimedes	(287-212 B.C)		Bose (1897-1974)
Galileo	(1564-1642)		Euler (1707 – 1783)
Coulomb	(1736-1806)		Coriolis (1792-1843)
Foucault	(1819-1868)		Bernoulli (1667-1748)

Mechanics: It a branch of physical sciences concerned with the state of rest or motion of bodies that are subjected to the action of forces.



Statics: Deals with the equilibrium of bodies that are either at rest or in motion with constant velocity.

Dynamics is concerned with the accelerated motion of bodies. It is further divided into

(i) **Kinematics** deals the study of motion of bodies without reference to force. It deals only with displacement, velocity, acceleration as a function of time and their relations.

(ii) **Kinetics** deals with the study of motion of bodies and the forces which are responsible for the motion.

Fundamental Concepts :

Basic quantities:

(i) Length (L): Length is a measurement to locate the position of a point in space and there by describe the size of physical system. Once a standard unit of length is defined, the geometric properties of a body can be defined as multiples of the unit length.

(ii) Time (T): Time is conceived as succession of events. Time plays important role in the study of dynamics.

(iii) Mass: Mass is a property of matter by which the action of one body with that of another can be compared. Simply, Mass is the quantity of matter possessed by a body. It is also quantitative measure of inertia of the body.

(iv) Force: It is an agency which changes or tends to change the state of rest or of uniform motion of a body. A force is completely characterized by its magnitude, direction and point of application.

Idealizations: Idealizations are the models that are used in Mechanics in order to simplify the application of the theory.

Particle: A particle has a mass but its shape and size can be neglected.

Ex: Size of earth is insignificant compared to the size of its orbit and here earth can be modeled as a particle when studying its orbital motion.

Rigid Body: Rigid body is that which does not deform under the action of applied forces. Though the physical bodies deform slightly under the action of loads on external forces, this deformation is neglected while studying mechanics of rigid bodies.

Continuum: Whether they may be solids are fluids are always idealized to be in continuum i.e they have a continues distribution of mass with no voids, empty spaces .

Weight: In case of a particle located at or near the surface of earth the only gravitational force having any sizable magnitude is that between the earth and the particle. This force is termed as weight.

$$W = G\,\frac{m\,m_2}{r^2} \qquad \text{let } g = G\,\frac{m_2}{r^2}$$

$$\therefore \qquad W = m.g$$

Space: It is a region extending in all directions. Position of a particle in space is w.r.t some reference system by linear or angular measurements.

Matter: It is the substance which occupies the space.

Inertia: It is the property of matter causing resistance to change in the state of rest or motion of a body.

Scalar: Scalar quantities are those with which a magnitude is only associated

Ex: Time, Volume, Density, Speed, Energy, Mass.

Vector: Vector quantities are those with which both magnitude as well as direction are associated.

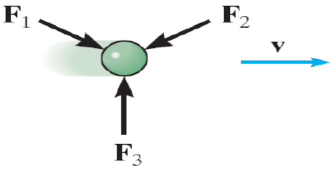
Ex: Displacement, Velocity, Acceleration, Force, Moment, Momentum, Weight.

Basic laws of Mechanics:

Newton's first law:

Each body remains in its state of rest or motion uniform in direction until it is made to change this state by imposed forces.

First law contains the principle of the equilibrium of forces → main topic of concern in Statics



Equilibrium

Newton's second law:

The change of motion is proportional to the imposed driving force and occurs along a straight line in which the force acts.

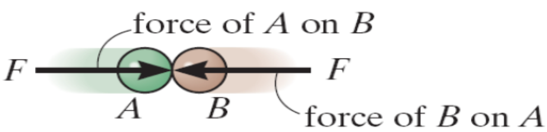
Accelerated motion

F = ma

Second Law forms the basis for most of the analysis in Dynamics

Newton's third law:

To every action there is always an equal reaction: or the mutual interactions of two bodies are always equal but directed contrary.



Action – reaction

Third law is basic to our understanding of Force → Forces always occur in pairs of equal and opposite forces.

Newton's Law of Gravitational Attraction:

This law governs the gravitational attraction between any two particles and stated mathematically.

$$F = G\frac{m_1 m_2}{r^2}$$

Where, F – Force of gravitation between two partials

G – Gravitational constant = 66.73 (10^{-12}) m^3 / (kg-s^2)

m_1, m_2 – Mass of each particle

r - Distance between the two particles.

Multiples				**Sub multiples**		
Tera	T	-	10^{12}	Pico -	p -	10^{-12}
Giga	G	-	10^{9}	Nano -	n -	10^{-9}
Mega	M	-	10^{6}	Micro-	µ -	10^{-6}
Kilo	K	-	10^{3}	Milli -	m -	10^{-3}
Hecto	A	-	10^{2}	Centi-	c -	10^{-2}

Deca C - 10^1 Deci - d - 10^{-1}

SI Unit: The international system of units, abbreviated SI after the French "Systeme International d' 'unites'.

In SI units: Length in metres (m)

Time in seconds (s)

Mass in kilograms (kg)

Unit of force is called a Newton (N) which is a derived from F = ma

Thus one Newton is equal to a force required to give 1 kg of mass producing an acceleration of $1 m/s^2$

($N = kg\text{-}m/s^2$)

If weight of a body located at the standard location, is to be determined in Newton's, then W=mg Here $g = 9.81\ m/s^2$

Therefore a body of mass 1 kg has a weight of 9.81 Newton

Similarly, 2kg mass has a weight of (2x9.81) = 19.62 N

Fundamental (Base) units:

Physical quantity		Base unit	Unit symbol	Dimension
1.	Length	metre	m	L
2.	Mass	kilogram	kg	M
3.	Time	second	s	T
4.	Temperature	Kelvin	k	t
5.	Current	Ampere	A	I
6.	Luminous Intensity	Candela	Cd	
7.	Amount of substance	mole	mol	

Supplementary units:

8.	Plane angle	radian	rad
9.	Solid angle	steradian	sr

1 Pascal (Pa) = $1\ N/m^2$

MPa = Mega pascal = 10^6 Pa = $\frac{10^6 \text{ x N}}{\text{m}^2}$ or MPa = $\frac{\text{N}}{\text{mm}^2}$

Derived Units:

Force	Newton	= kg-m/s^2
Energy, work or heat	Joule	= Nm = kg-m^2/s^2
Power	watt	= Joules / second = $\frac{J}{s} = \frac{Nm}{s} = \frac{kg-m^2}{s^3}$
Pressure or stress	Pascal	$= \frac{N}{m^2} = \frac{kg}{m\,s^2}$
Frequency	hertz	= HZ = s^{-1}

Some derived units:

Acceleration	m/s^2
Angular Acceleration	rad/s^2
Angular displacement	rad
Angular Momentum	kg m^2/s
Angular Velocity	rad /s
Area	m^2
Couple, moment	N- m
Density	kg/m^3
Discharge	m^3/s
Energy	J (N-m)
Force	N (kg-m/s^2)
Frequency	HZ ($\frac{1}{s}$)
Modulus of Elasticity	N/m^2 (Pa)
Momentum	Kg m/s (Ns)
Moment of Inertia	kg-m^2 or m^4
Power	watt (N-m/s)
Pressure, Stress	Pa (N/m^2)
Specific Energy	J /kg
Specific Volume	kg/m^3

Speed, Velocity	m/s
Torque	N-m
Viscosity	Ns/m^2
Volume	m^3
Weight	Newton ($kg\text{-}m/s^2$)
Work	Joule (N-m)

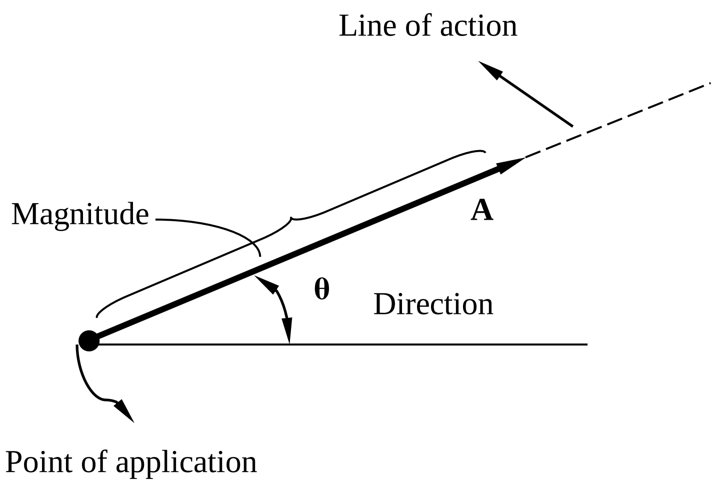
Forces And Examples:

Force: It is defined as an agency which changes or tends to change the position of body at rest or motion of a body. It has the capacity to produce motion to a body.

EX: Push, pull, shear, torque, accelerating force, gravitational force, frictional force, magnetic force etc.

Force is completely defined only when the following four characteristics are specified.

(i) Magnitude (ii) Point of application (iii) Line of action (iv) direction

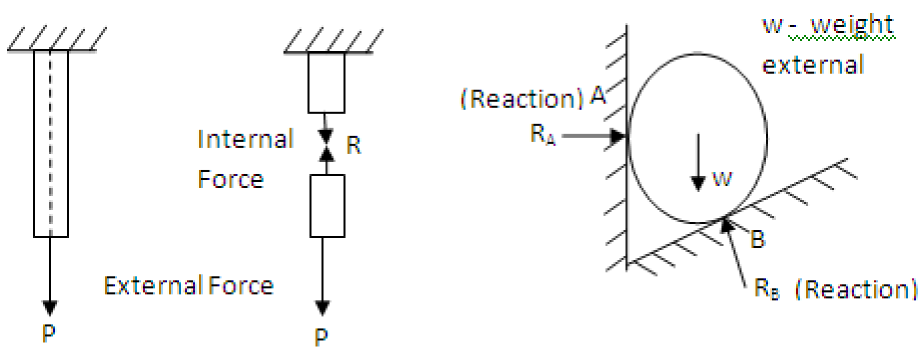


Types of forces:

External forces: Applied forces are external forces acting on a body and also weight of the body.

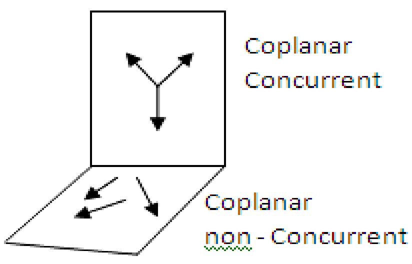
Reactions: When a body is subjected to external forces, the body resists by its reactions known as reactive forces.

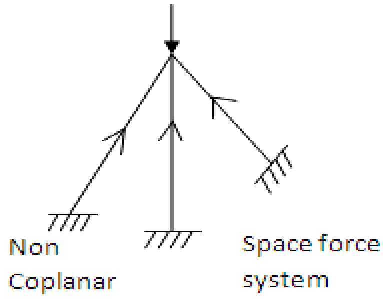
Internal forces: These are the forces induced in the members due to external forces.



<u>**Classification of Forces**</u>

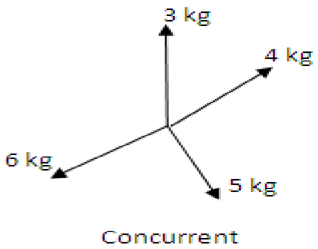
1. Coplanar Forces: When all the forces in a system lie in the same plane, they are co planar forces and such a system is known as coplanar force system.

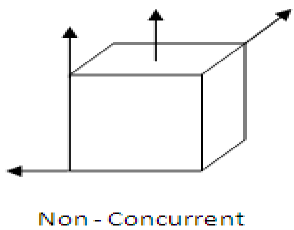




2. Non-Coplanar forces: When forces of a force system do not lie in the same plane, they are known as Non-Coplanar forces.

3. Concurrent forces: When line of action of all forces pass through a single point that force system is known as concurrent force system.





4. Non-concurrent force system: When the lines of forces of a system do not pass through a single point, the force system is known as Non-Concurrent force system.

Other combinations are

5. Coplanar – Concurrent: When line of action of all forces pass through a single point that force system also lie in the same plane is known as coplanar concurrent force system.

6. Coplanar – Non concurrent: The system of forces will act on same plane but they will not meet at same point.

7. Non-coplanar – Concurrent: The system of forces will not act on same plane but they will meet at same point.

8. Non-coplanar – Non concurrent: The system of forces will not act on same plane but they will not meet at same point.

9. Parallel force systems: When the forces in a system lie parallel to each other, that system is known as parallel force system.

10. Collinear force system: If the line of action of all the forces in a system lies along a single line, then it is called a collinear force system.

Basic Principles of a force:

- **Principle of physical independence of force:**

Action of force on bodies are independent, in other words the action of forces on a body is not influenced by the action of any other force on the body.

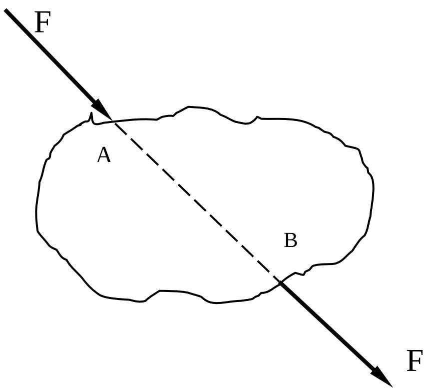
- **Principle of transmissibility of forces:**

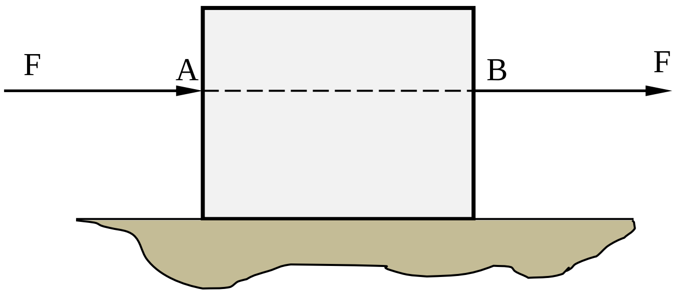
The point of application of force on a rigid body can be changed along the same line of action maintaining same magnitude and direction without affecting the effect of the force on the body .

(or)

According to the principle of Transmissibility of a force, the state of rest or motion of body is unaltered, if a force acting on a body is replaced by another force of same magnitude and direction but acting at any point on the body along the line of the replaced force.

Force at A = Force at B





Addition of Forces:

Addition of (Forces) by Head to Tail Rule

To add two or more than two vectors (forces), join the head of the first vector with the tail of second vector, and join the head of the second vector with the tail of the third vector and so on. Then the resultant vector is obtained by joining the tail of the first vector with the head of the last vector. The magnitude and the direction of the resultant vector (Force) are found graphically and analytically.

Resultant Force:

A resultant force is a single equivalent force, which produce same affect on the body as that of all the given forces.

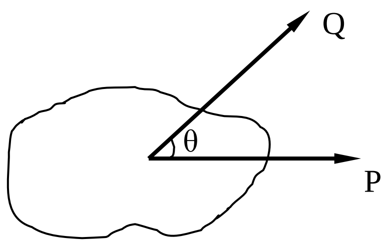
Composition of Forces:

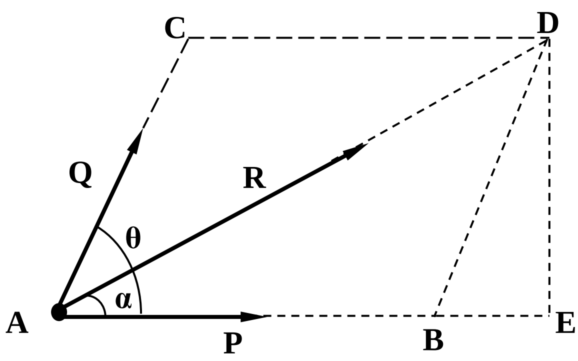
The process of finding out the resultant Force of given forces (components vector) is called composition of forces. A resultant force may be determined by following methods

1. Parallelogram laws of forces or method
2. Triangle law of forces or triangular method
3. Polygon law of forces or polygon method.
4. Resolving Method or Method of Projections

Parallelogram Law of forces:

If two forces acting simultaneously on a body at a point are represented in magnitude and direction by the two adjacent sides of a parallelogram their resultant is represented in magnitude and direction by the diagonal of the parallelogram which passes through the intersection of two sides representing the forces.





Let AB = force, P, AC = force, Q, AD = Resultant, R

Angle between P and Q forces = θ

$AD^2 = AE^2 + DE^2$

$AD^2 = (AB + BE)^2 + DE^2 = AB^2 + BE^2 + 2\,AB.BE + DE^2$

$$AD^2 = AB^2 + BE^2 + DE^2 + 2\ AB.BE$$

$$= AB^2 + BD^2 + 2\ AB.BD\cos\theta$$

$$= AB^2 + AC^2 + 2\ AB.AC\cos\theta \quad \text{(since BD = AC)}$$

$$\therefore R^2 = P^2 + Q^2 + 2PQ\cos\theta$$

$$\tan\alpha = \frac{DE}{AE} = \frac{BD\sin\theta}{AB+BE} = \frac{BD\sin\theta}{AB+BD\cos\theta} \quad \text{(since BD = AC)}$$

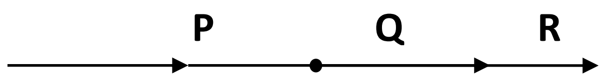
$$\tan\alpha = \frac{AC\sin\theta}{AB+AC\cos\theta}$$

$$\therefore \tan\alpha = \frac{Q\sin\theta}{P+Q\cos\theta}$$

Special cases:

Case-I If $\theta = 0^0$, $\cos 0° = 1.$

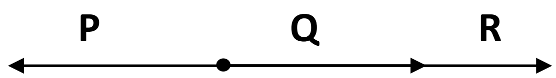
$$R = \sqrt{P^2+Q^2+2PQ} = \sqrt{(P+Q)^2} = P+Q$$



Case-II If $\theta = 180^0$, $\cos 180° = -1.$

$$R = \sqrt{P^2+Q^2-2PQ} = \sqrt{(P-Q)^2} = P-Q$$

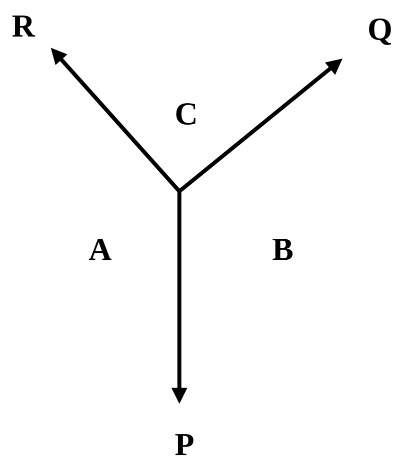
Case-III If $\theta = 90^0$

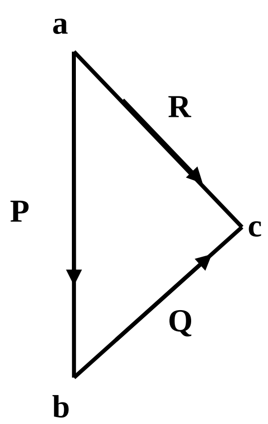


$$R = \sqrt{P^2+Q^2+2PQ\cos 90^0} = R = \sqrt{P^2+Q^2-0} = \sqrt{P^2+Q^2}$$

<u>Triangle Law of forces</u>:

This Law states that "If two forces acting at a point are represented both in magnitude and direction by the two sides of a triangle, taken in order, the third side of the triangle, taken in opposite order represents the resultant of the two forces in magnitude and direction".

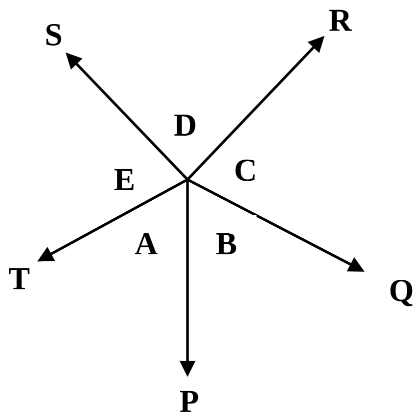




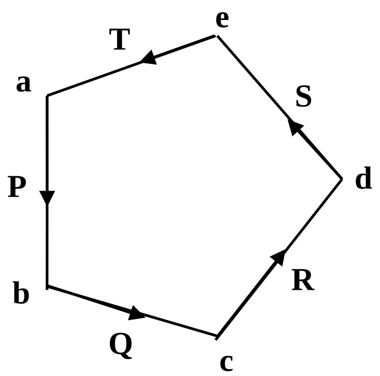
Space Diagram **Vector Diagram**

<u>Polygon Law of forces</u>:

This law states that “If a number of concurrent forces acting simultaneously on a body and are represented in magnitude and direction by the sides of a polygon taken in order, then the resultant is represented in magnitude and direction by the closing side of polygon, taken in opposite order”.



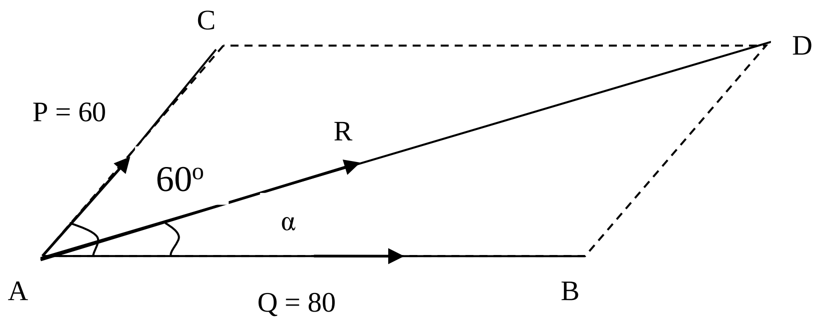
Space Diagram



Vector Diagram

Ex: Two forces 80N and 60N act at an angle of 60^0. Determine the magnitude and direction of the resultant.

Sol:



$$R = \sqrt{P^2 + Q^2 + 2PQ\,cos\theta}$$

$$R = \sqrt{80^2 + 60^2 + 2\,(80)\,(60)\,cos\theta}$$

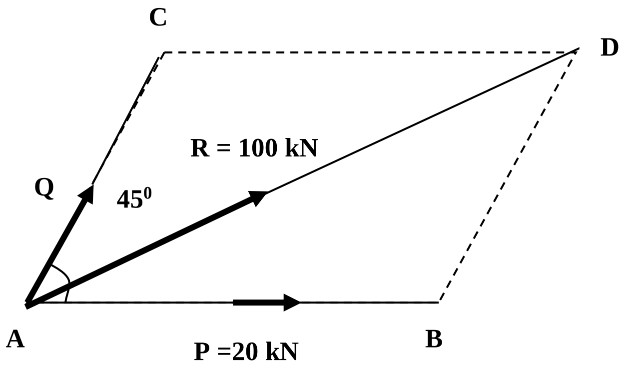
$$R = \sqrt{6400 + 3600 + 4800}$$

$$R = \sqrt{148000} \quad = 121.66 \text{ N}$$

$$\tan\alpha = \frac{\theta \sin\theta}{P + Qcos\theta} \quad = \frac{60\;x\,\sin 60^0}{80 + 60\;cos\;60^0}$$

$$\tan\alpha = \frac{30\sqrt{3}}{110} = 0.4723 \qquad \therefore \alpha = 25^{o}\text{-}17^{'}$$

Ex: Two coplanar concurrent forces act towards a point with an angle of 45^0 between there. If their resultant is 100kN, and one of the forces is 20kN, calculate the other force.



Sol: Given ,

P=20 kN, R=100kN and $\theta = 45^0$

$R^2 = P^2 + Q^2 + 2PQ\cos\theta$

$100^2 = 20^2 + Q^2 + 2 \times 20 \times Q\cos 45^0$

$100^2 = 20^2 + Q^2 + 28.284\,Q$

$Q^2 + 28.284Q = 10000 - 400 = 9600$

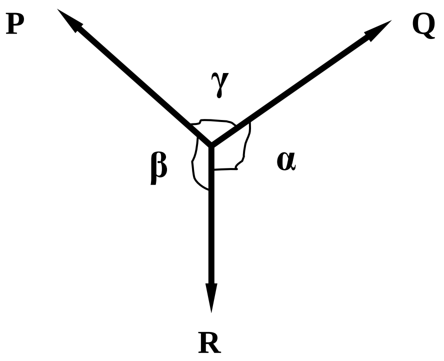
$(Q+14.142)^2 = 9600 + 14.14^2 = 9800$

$Q+14.142 = 98.995$

$Q = 98.995 - 14.142 = 84.853kN$

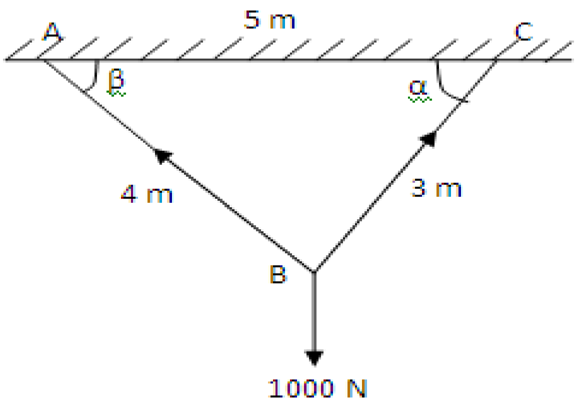
<u>Lami's theorem</u>:

Statement: "If a body is in equilibrium under the action of three forces, each force is proportional to the sine of angle between the other two forces".



$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Ex: A body of weight 1000N is suspended by two strings of 4m and 3m lengths attached at the same horizontal level 5m apart. Calculate the forces in the strings.



Since the triangle is with 3, 4, 5 m sides, angle at B = 90^0

$$\alpha + \beta = 90^0$$

$$\sin\alpha = \frac{4}{5} = 0.8 \ ; \quad \sin\beta = \frac{3}{5} = 0.6$$

Applying Lami's theorem,

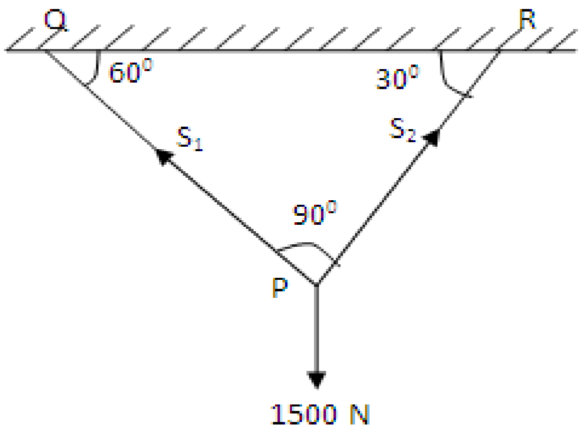
$$\frac{1000}{sin90^0} = \frac{P}{\sin(180-\beta)} = \frac{Q}{\sin(180-\alpha)}$$

$$\frac{1000}{1} = \frac{P}{\sin\beta} = \frac{Q}{\sin\alpha}$$

$$\therefore P = 1000 \times 0.6 = 600 \text{ N}$$

$$\therefore Q = 1000 \times 0.8 = 800 \text{ N}$$

Ex: Calculate the forces in members PQ and PR for the frame shown in fig. using Lami's theorem.



$$\frac{1500}{sin90^0} = \frac{S_1}{sin120^0} = \frac{S_2}{sin150^0}$$

$$\frac{1500}{1} = \frac{S_1}{sin60^0} = \frac{S_2}{sin30^0}$$

$$\therefore S_1 = 1500 \times 0.866 = 1299\text{N}$$

$$\therefore S_2 = 1500 \times 0.5 = 750\text{N}$$

Free Body Diagram

The diagram of a body isolated from all other supports and considering only the forces like weight, reactions, applied forces etc, acting on the body it known as "Free Body Diagram" (F.B.D).

The FBD makes it easy to apply the laws of equilibrium to the set of forces acting on a body and it helps in solving complicated problems.

Procedure of drawing Free Body Diagram

To construct a free-body diagram, the following steps are necessary:

- Draw Outline Shape

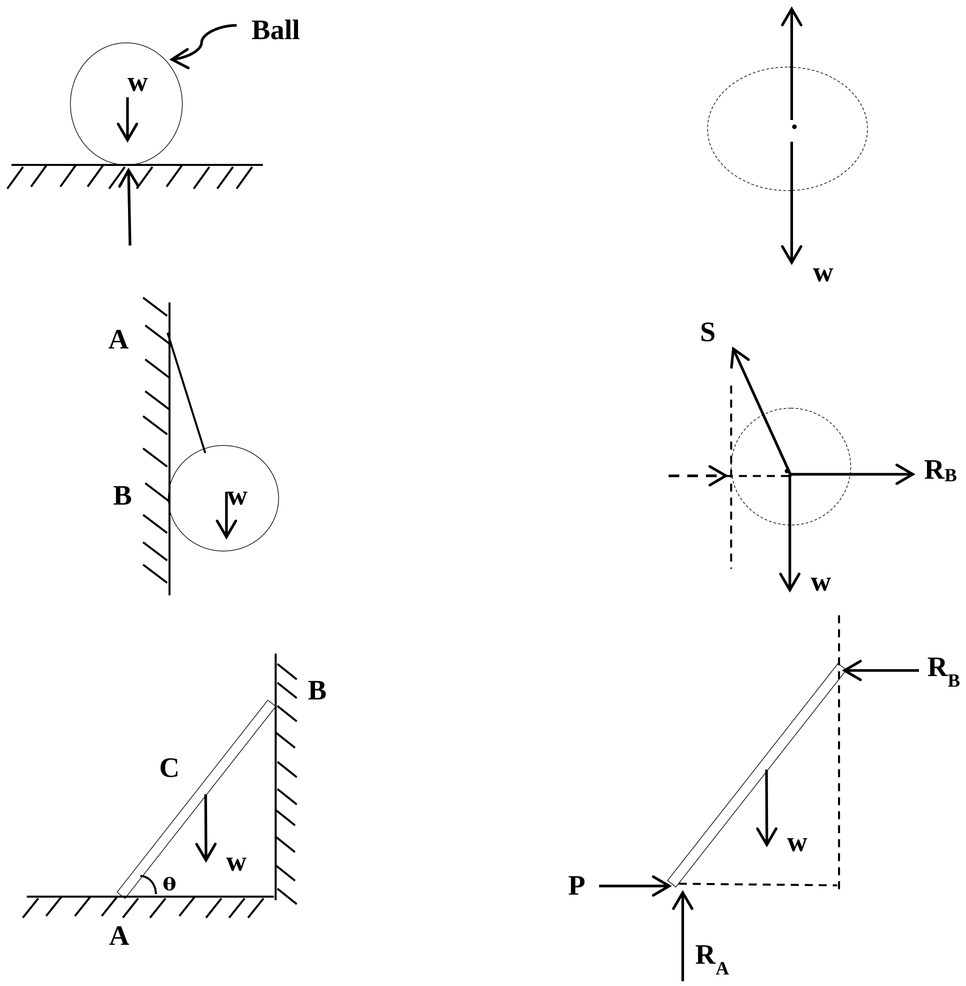
Imagine that the particle is cut free from its surroundings or isolated by drawing the outline shape of the particle or body only

- Show All known Forces

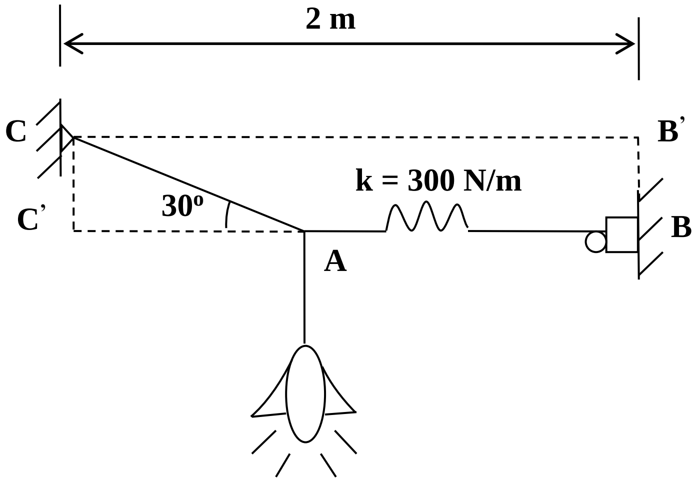
Show on this sketch all the forces acting on the particle. There are two classes of forces that act on the particle. They can be active forces, which tend to set the particle in motion, or they can be reactive forces which are the results of the constraints or supports that tend to prevent motion.

- Identify Each unknown Force

The forces that are known should be labelled complete with their magnitudes and directions. Letters are used to represent the magnitudes and directions of forces that are not known.



Ex: Determine the required length of rod AC as in figure. So, that the 8kg lamp is suspended in the position shown. The undeformed length of spring AB is L = 0.4 m and the spring has a stiffen K_{AB} = 300N/m.



Sol: In spring,

$F = k\,.s$

$W = 8 \times 9.81$

$W = 78.5$ N

$K = 300$ N/m

Making $\Sigma F_x = 0$; $T_{AB} - T_{AC}.\cos 30^0 = 0$

Making $\Sigma F_y = 0$; $T_{AC}.\text{Sin}30^0 - 78.5 = 0$

$\therefore\ T_{AC} = \frac{78.5}{0.5} = 157.0$ N

$T_{AB} = 157 \times 0.866 = 136.0$ N

The stretch in spring = Stretched length - Un Stretched length

$S_{AB} = l_{AB} - l'_{AB}$

$T_{AB} = k\,.\,S_{AB}$

$\therefore\ S_{AB} = \frac{136}{300} = 0.453$ m

$\therefore l_{AB} = 0.453 + 0.4 = 0.853$ m

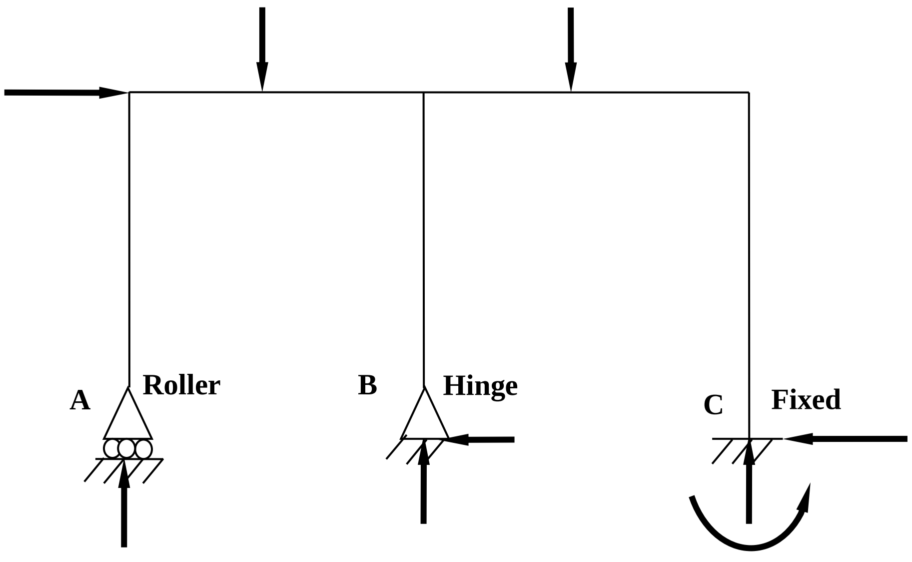
CB' = 2 m

$\therefore$ AC' = 2 - 0.853 = 1.147 m

$l_{AC} \cos 30^0 = 1.147$

$\therefore l_{AC} = \frac{1.147}{\cos 30^0} = \frac{1.147}{0.866} = 1.324$ m

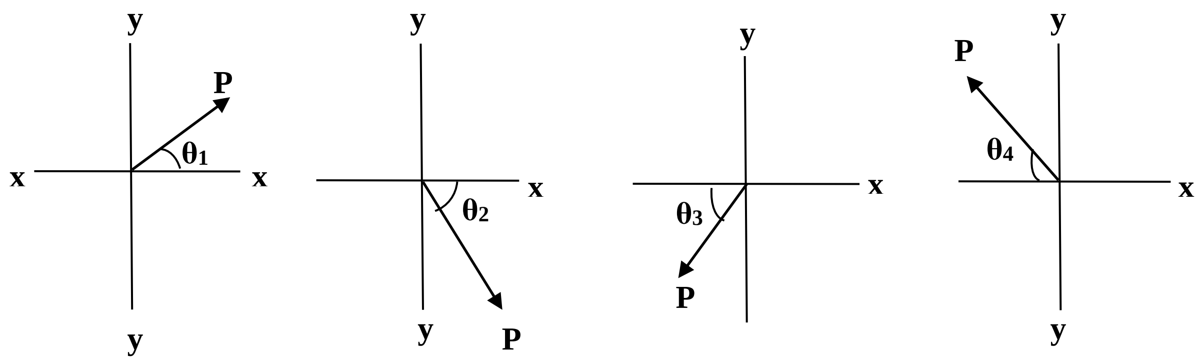
Support Conditions:



Displacement: x = ✓	x = 0	x = 0
Displacement: y = 0	y = 0	y = 0
Rotation: ɵ = ✓	ɵ = ✓	ɵ = 0
Force: X = 0	X = ✓	X = ✓
Force: Y = ✓	Y = ✓	Y = ✓
Moment: M = 0	M = 0	M = ✓

Resolution of forces:

As a number of coplanar and concurrent forces can be combined into a single force; similarly, a single force can be split into a number of forces in the required directions. This process is called “Resolution of forces”.



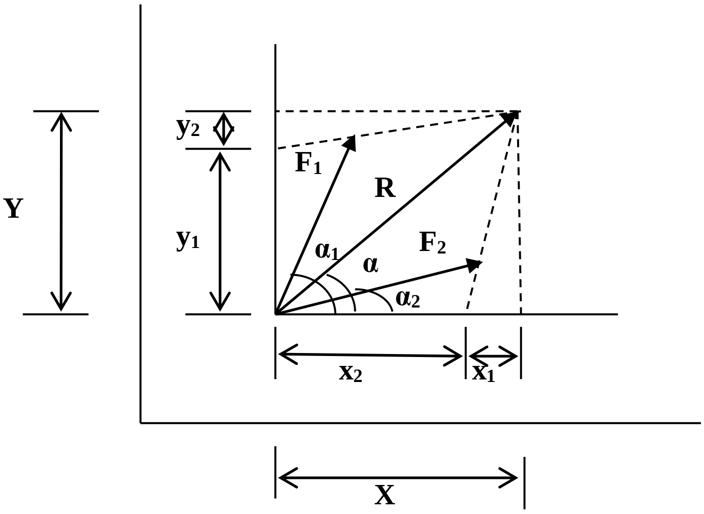
$x_1 = + P_1\cos\theta_1$	$x_2 = + P_2\cos\theta_2$	$x_3 = - P_3\cos\theta_3$	$x_4 = - P_4\cos\theta_4$
$y_1 = + P_1\sin\theta_1$	$y_2 = - P_2\sin\theta_2$	$y_3 = - P_3\sin\theta_3$	$y_4 = + P_4\sin\theta_4$

Resolution:

The replacement of a single force by several components which will be equivalent in action to the given force is called the problem of "Resolution of a force".

Let

F_1, F_2 – Two forces, R – Resultant, α_1, α_2 – Angles of F_1 and F_2 w.r.t x-x axis.



$x_1 = F_1\cos\alpha_1,\ x_2 = F_2\cos\alpha_2$

$y_1 = F_1\sin\alpha_1,\ y_2 = F_2\sin\alpha_2$

$x = R\cos\alpha,\ y = R\sin\alpha$

$x^2+y^2 = R^2(\cos^2\alpha+\sin^2\alpha) = R^2$

But $x = x_1 + x_2 = \sum x$

$y = y_1 + y_2 = \sum y$

$R = \sqrt{X^2+Y^2} = \sqrt{\sum X^2 + \sum Y^2}$

$\tan\alpha = \frac{\sum Y}{\sum X} = \frac{Y}{X}$

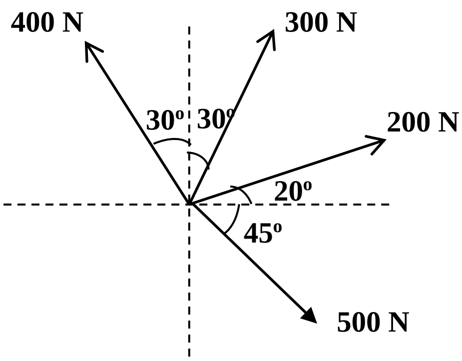
When $\sum x = 0$, $\sum y = Q$, its resultant is a vertical force.

When $\sum y = 0$, $\sum x = P$, its resultant is a horizontal force.

When $\sum x = 0$, $\sum y = 0$, its resultant is zero. The system is in equilibrium condition.

$\sum x = 0$, $\sum y = 0$ are equations of equilibrium.

Ex: The following are the four forces acting at a point on a body. Find the resultant and its position.



i. 300N at N 30° E
ii. 200N at N 70° E
iii. 500N at S 45° E
iv. 400N at N 30° W

Sol:

$\sum x = +300\cos60^\circ + 200\cos20^\circ + 500\cos45^\circ - 400\cos60^\circ$

$= +150.00 + 187.50 + 353.55 - 200$

$= +491.5$ N

$\sum y = +300\sin60^\circ + 200\sin20^\circ - 500\sin45^\circ + 400\sin60^\circ$

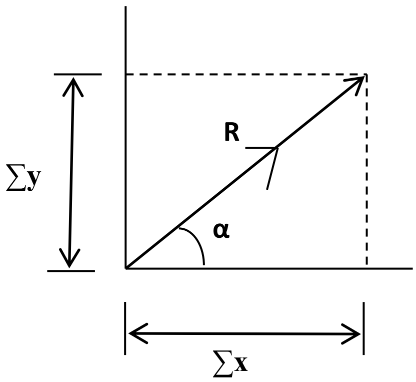
$= +259.81 + 68.40 - 353.55 + 346.41$

$= +321.07$ N

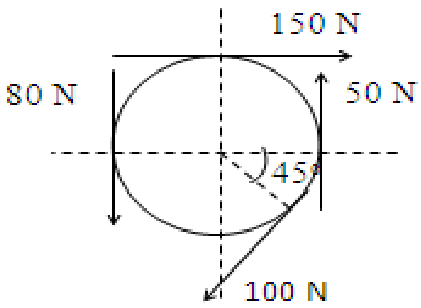
$R = \sqrt{\sum X^2 + \sum Y^2} = \sqrt{491.5^2 + 321.07^2}$

R = 587.08 N

$\alpha = \tan^{-1}(\frac{321.07}{491.5}) = 33^0 15'$



Ex: Determine the resultant of the four forces tangent to the circle of radius 3 m as shown in fig.



Sol: $\sum x = +150 - 100 \cos 45^o$

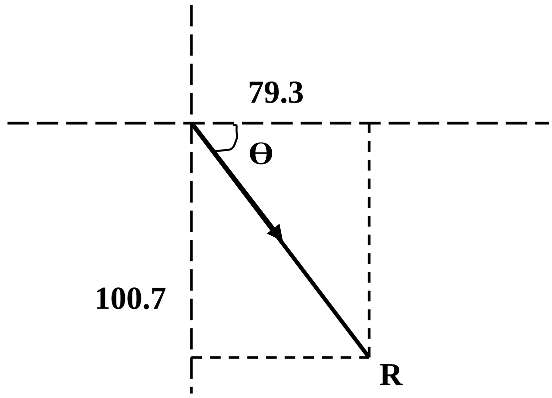
$\sum x = +150 - 70.7 = +79.3$ N

$\sum y = +50 - 80 - 100 \sin 45^o$

$= -30 - 70.7 = -100.7$ N

$R = \sqrt{79.3^2 + 100.7^2} = 128.17$ N

$\theta = \tan^{-1}(\frac{100.7}{79.3}) = 51^0 78'$



<u>Moment of a force:</u>

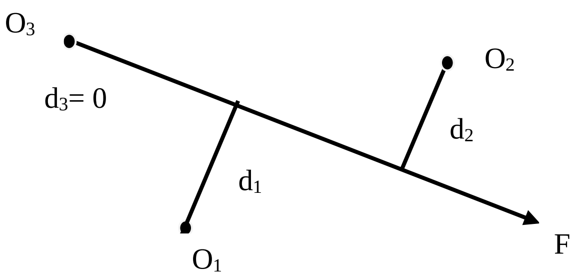
Moment of a force is the turning effect of a force about a point. It is also a measure of rotational effect.

Moment of a force is defined as the product of the magnitude of the force and the perpendicular distance of the point from the line of action of the force.

M = F. d where d is perpendicular distance of point from the line of action of the force.

Taking moments about O_1

$M_1 = F.d_1$ (clock wise)



Taking moments about O_2

$M_2 = F.d_2$ (Anti clock wise)

Taking moments about O_3

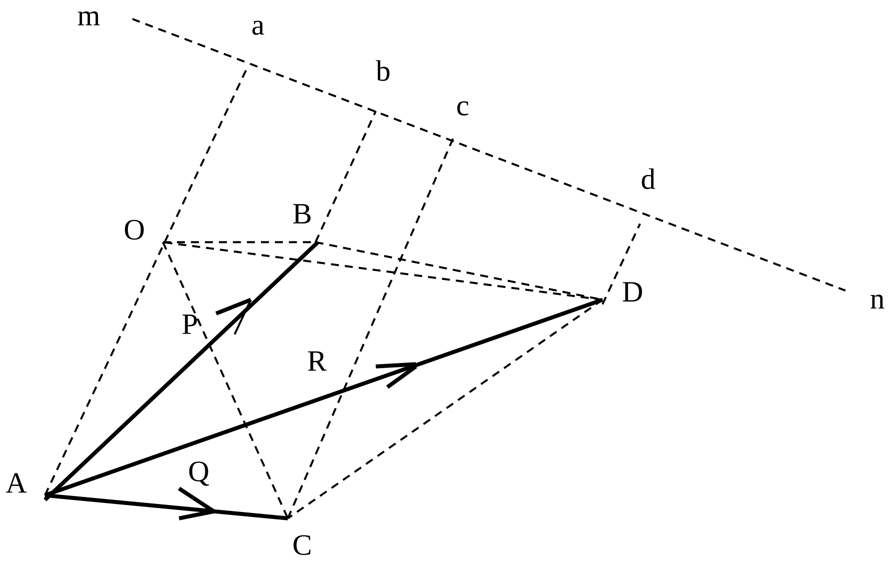
$M_3 = F.d_3$, $M_3 = 0$ (Since $d_3 = 0$)

The points about which the moments are considered is the moment center (O_1, O_2 etc).

If the moment center is on the line of force then the moment of force about that force is zero.

Varignon Theorem:

It states that the algebraic sum of the moments of a system of coplanar forces about a moment center in their plane is equal to the moment of their resultant about the same moment center.



Let P, Q be the two forces acting at A, R be the resultant and O be the moment center.

In the plane of force, take any line mn perpendicular to the line OA, joining the moment center with the concurrent point of forces P & Q.

Construct perpendicular Aa, Bb, Cc, Dd as shown in the figure.

Area of triangle $\triangle$ OAB = $\frac{1}{2}$.OA. ab

Area of triangle $\triangle$ OAC = $\frac{1}{2}$.OA. ac

Area of triangle $\triangle$ OAD = $\frac{1}{2}$.OA. ad

Since $\quad ad = ab + bd \quad (bd = ac)$

$ad = ab + ac$

Therefore we conclude that

Area of triangle OAD = Area of triangle OAB + Area of triangle OAC

$$\frac{1}{2}.OA.\ ad \quad = \quad \frac{1}{2}.OA.\ ab + \frac{1}{2}.OA.\ ac$$

Multiply with 2 on both sides

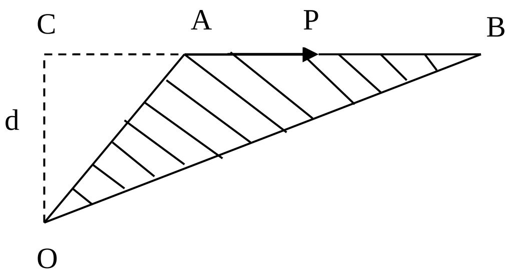
2 × Area of triangle OAD = 2 × Area of triangle OAB + 2 × Area of triangle OAC

$$OA.\ ad \quad = \quad OA.\ ab + OA.\ Ac$$

Moment of resultant about O = moment of P about O + moment of Q about O

Hence the theorem is proved.

<u>Graphical representation of a force</u>



Let AB be the vector of force P

Consider $\triangle$ OAB in which AB as base and O (moment centre) be apex of the triangle.

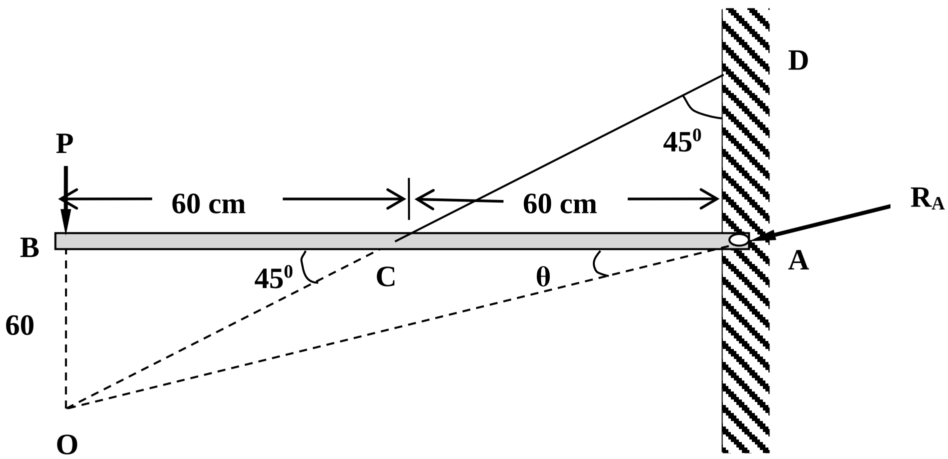
Area of the triangle OAB=$\frac{1}{2}$AB(d)

O be the moment center

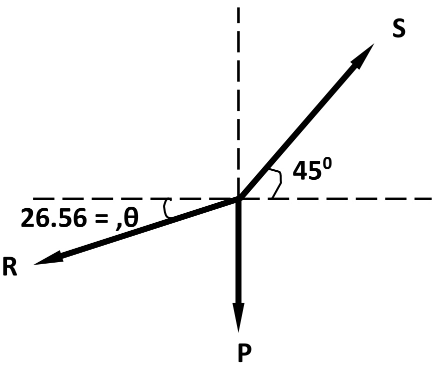
But, Moment of force =AB(d)

Therefore moment of force = Twice the area of triangle formed with force vector as base and moment center as vertex.

Ex: A horizontal beam AB is hinged to a vertical wall at A and supported at its midpoint C by a tie rod CD as shown in figure. Find the tension S in the tie rod and reaction at A due to a vertical load applied at B.



Sol:

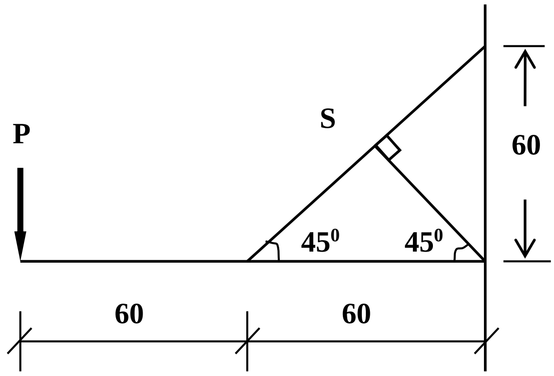


Tan θ = $\frac{60}{120}$ = 0.50

$\theta = 26^{0}.565$

$$\frac{nhy6P}{Sin\ 165.565^{0}} = \frac{S}{Sin\ 63.435^{0}} = \frac{R_A}{Sin\ 135^{0}}$$

$$\frac{P}{Sin\ 18.435^{0}} = \frac{S}{Sin\ 63.435^{0}} = \frac{R_A}{Sin\ 45^{0}}$$



$$S = P \times \frac{Sin\ 63.435^{0}}{Sin\ 18.435^{0}} = \frac{0.8944}{0.3162} P = 2.83P$$

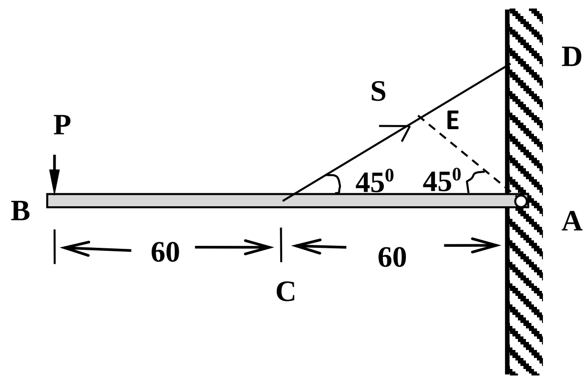
$$R_A = \frac{P\ Sin\ 45^{0}}{Sin\ 18.435^{0}} = \frac{0.7071}{0.3162} P = 2.236\ P$$

Applying moment equation,

Taking moments about A

$$P \times 120 = S \times \frac{60}{\sqrt{2}}$$

$$S = \frac{120 \times \sqrt{2}}{60} = 2\sqrt{2}\ P \qquad = 2.83\ P$$



(Or)

Using method of moments

Taking moments about A,

$S \times 60 \times 0.707 = 120 \times P$

$$S = \frac{2}{0.707} P = 2.83\ P$$

Ex: A rigid bar AB is supported in a vertical plane and carries a load Q at its free end as shown in figure. Neglecting the weight of the bar itself, compute the magnitude of the tensile force S induced in the horizontal string CD.

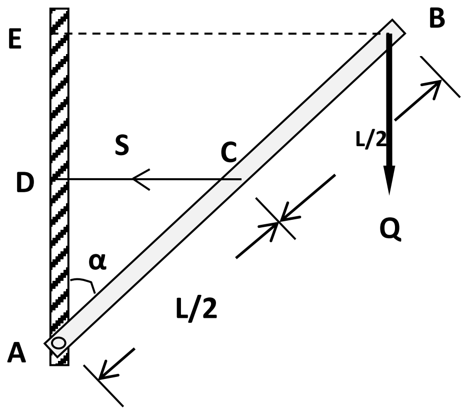
Sol: Taking moments about A

Sum of clockwise moments = sum of anti clockwise moments

$$S \times AD = Q \times BE$$

$$S \times \frac{L}{2}\cos\alpha = Q \times L\sin\alpha$$

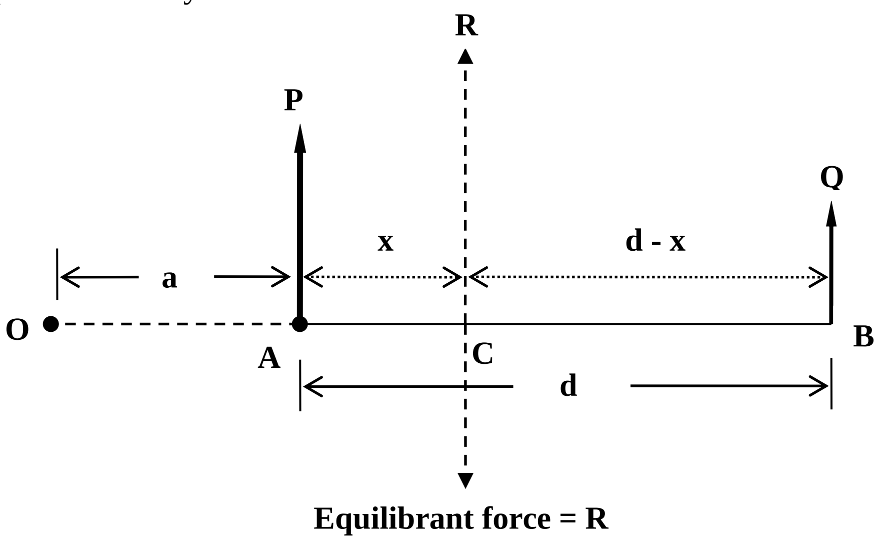
$$\boxed{S = 2\,Q\tan\alpha}$$



Parallel Forces:

A system of forces which are parallel to each other irrespective of their magnitude and direction is known as a parallel force system.

Like parallel force system: A system with two parallel forces acting in the same direction is known as like parallel force system.



Resultant of a like two parallel force system

$$R = P + Q$$

The position of resultant from any force is proportional to its distance from other force. R lies between two forces and nearer to the larger force.

Taking moments about A,

$$x \times R = Q \times d$$

$$\boxed{\frac{x}{d} = \frac{Q}{R}}$$

Taking moments about B

$$(d - x)\, R = P \,.\, d$$

$$\boxed{\frac{d-x}{d} = \frac{P}{R}}$$

Taking moments about O,

$$(a + x)\, R = a \,.\, P + (a + d)\, Q$$

$$(a + x)(P + Q) = P \,.\, a + Q \,.\, a + Q \,.\, d$$

$$\cancel{P\,a} + P\,x + \cancel{Q\,a} + Q\,x = \cancel{P\,a} + \cancel{Q\,a} + Q\,d$$

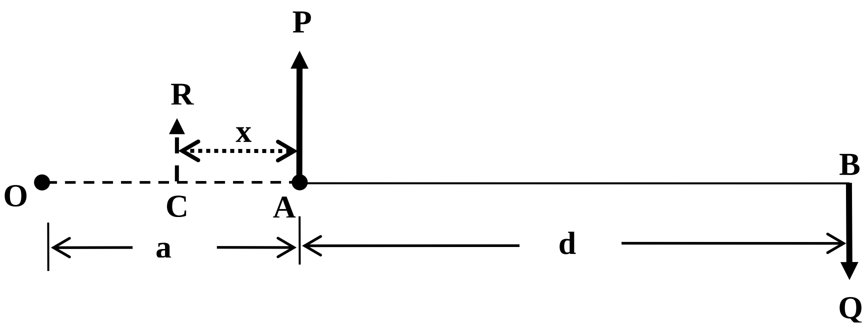
$$(P + Q)\, x = Q \,.\, d$$

$$\boxed{\frac{x}{d} = \frac{Q}{R}}$$

Unlike parallel forces:

A system of two forces with different magnitude but opposite in direction is known as an unlike parallel force system.

$$R = P - Q$$



The position of resultant from any force is proportional to its distance from other force. It lies outside the force system and nearer to the larger force.

Taking moments about A,

$$Q \times d = R \times x$$

$$\boxed{\frac{x}{d} = \frac{Q}{R}}$$

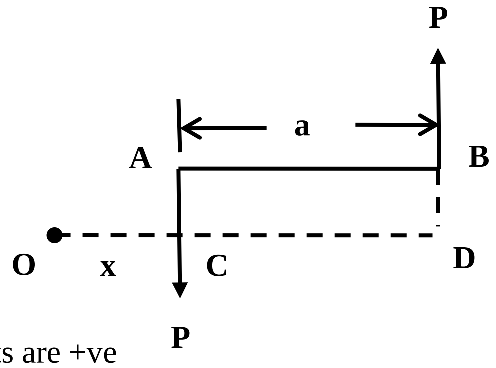
Taking moments about B

$$P \,.\, d = R\,(d + x)$$

$$\boxed{\frac{d+x}{d} = \frac{P}{R}}$$

Couple:

Two equal parallel forces acting in opposite direction, non collinear constitute a '*couple*'. The plane in which they act is called 'plane of couple'. The distance between their lines of action is the 'arm of couple'.



Moment of couple M = P.a (anti clock wise)

Moment of couple remains same irrespective of moment center.

Let O be the orbitary moment center

Taking moments about O and considering clockwise moments are +ve

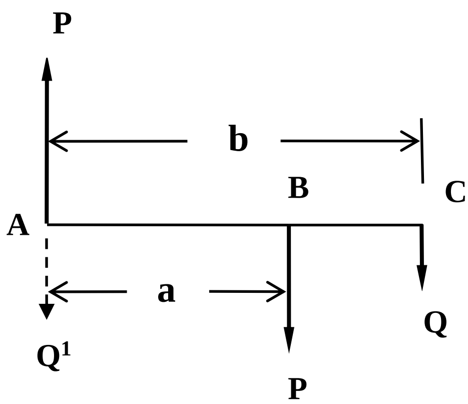
$$M = - P.\ OC + P.\ (OC + CD)$$

$$= - \cancel{P.\ x} + \cancel{P.\ x} + P.\ a$$

$$= P.a$$

Statement:

Action of couple on a body doesn't change if we change both magnitude and direction of forces and arm of the couple in such a way that the moment of the couple remains unchanged.



$M = P.\ a$

Resolve P at B into two parallel components

Q at C and Q^1 at A

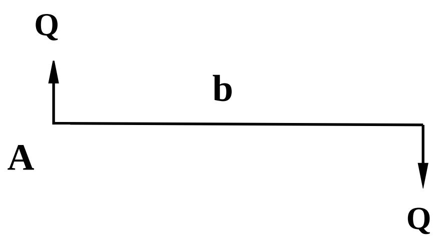
$Q = P.\ \frac{a}{b}$ and $Q^1 = P - Q$

When you substitute $Q^1 = P - Q$

Therefore moment of this couple

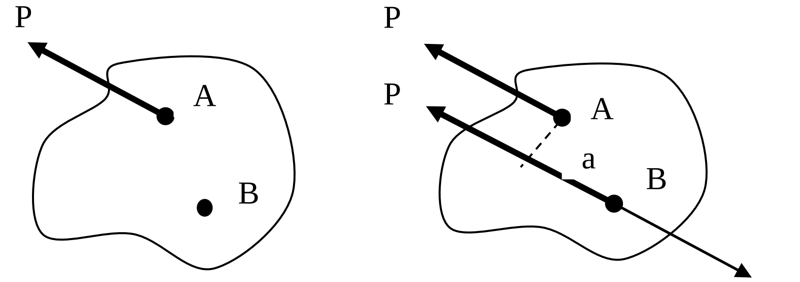
$M = Q.\ b$

$M = P.\ a = Q.\ b$



Statement:

A force at a point on a body can be replaced by an equal parallel force at other point on the same body together with a couple.



Ex: Find the resultant and its position for following figures.

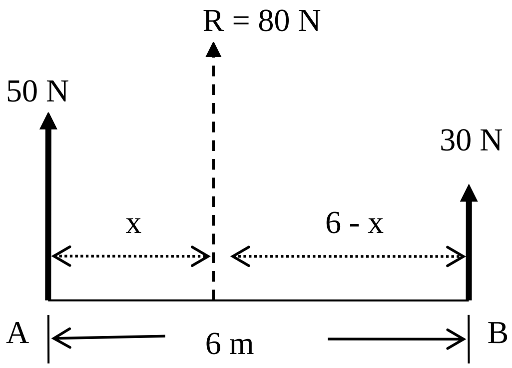


Fig A

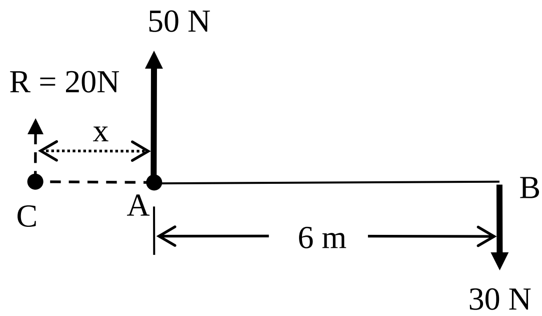


Fig B

Sol:

$R = 50 + 30 = 80$ N

$$\frac{x}{6} = \frac{30}{80} = \frac{Q}{R}$$

$$x = \frac{180}{80} = 2.25 \text{ m}$$

$6 - x = 3.75$ m

$R = 50 + 30 = 80$ N

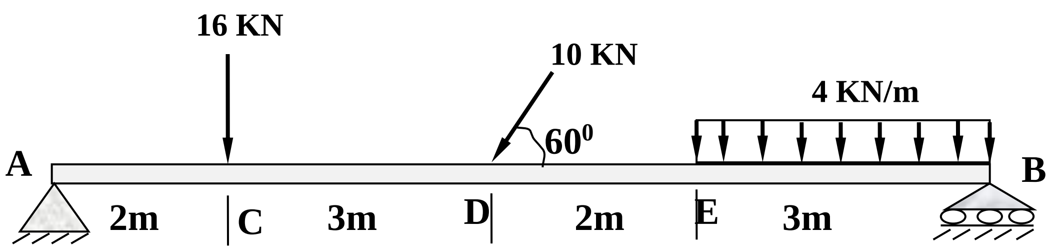
$$\frac{x}{6} = \frac{30}{20} = \frac{Q}{R}$$

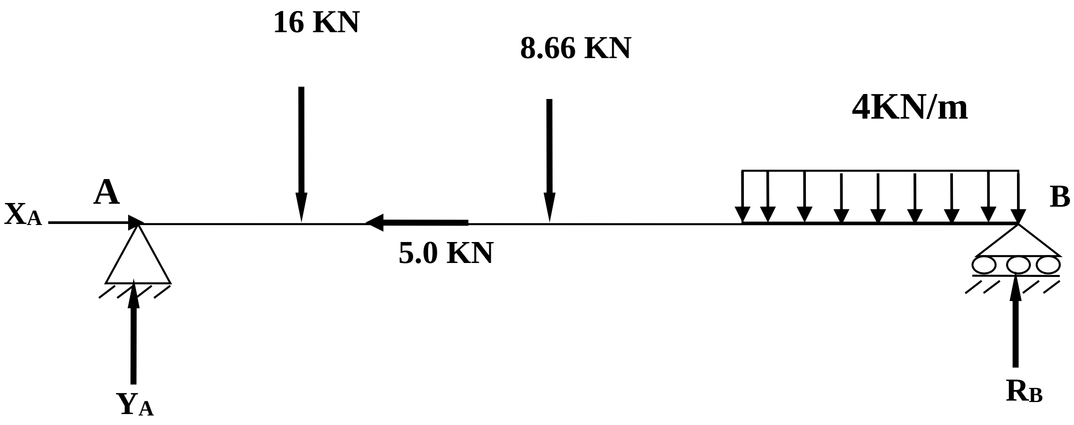
$$x = \frac{180}{20} = 9 \text{ m}$$

$$\frac{x+6}{6} = \frac{P}{R} = \frac{50}{20}$$

$x + 6 = 15 \rightarrow\rightarrow x = 9$ m

Ex: Determine the reactions at supports for the beam shown in fig.





Sol: Taking moment about A ,

$R_B \times 10 = (2\times16) + (5 \times 8.66) + 8.5\ (3\times4)$

$10\ R_B = 32 + 43.33 + 102$

$10\ R_B = 177.33$

$R_B = 17.73$ KN

$\Sigma\ x = 0 \rightarrow\rightarrow X_A = 5$ KN

$\Sigma y = 0 \rightarrow\rightarrow Y_A = 16 + 8.66 + 12 - 17.73$

$= 18.93$ KN

$R_A = \sqrt{5^2 + 18.93^2} = 19.58$ KN

$\Theta = \tan^{-1}\left(\frac{18.93}{5}\right) = 75.2^0.$

---THE END---

UNIT –II

Objective:

To understand the basic concepts of static and kinetic friction and apply to simple examples of bodies resting on horizontal and inclined planes, ladder.

Method of virtual work is an alternative approach to analyze the equilibrium of any loaded system and the unknown forces can be determined. For an equilibrium force system, the algebraic summation of work done by all effective force components is equals to zero. Virtual work principle can mainly applicable to simple beams, connected systems.

Syllabus:

Friction: Types of friction, Principle of friction, Coefficient of friction, Angle of friction, Angle of repose, Cone of static friction, Static and Dynamic friction, Limiting friction, Laws of friction – Simple problems.

Application of friction: Impending motion of connected bodies in horizontal and inclined planes, ladder friction, wedge friction – Simple problems.

Virtual work: Virtual displacement, Principle of virtual work. Applications to connected systems and reactions of simply supported beams.

Learning Outcomes:

Student will be able to

- Understand the concept of friction and its related terms and laws of static friction
- Draw the free body diagrams of bodies resting on rough horizontal and inclined planes at the point of impending motion
- Apply the friction concept to ladder examples
- Understand the concept of virtual work
- Explain the principle of virtual work through examples

Learning Material

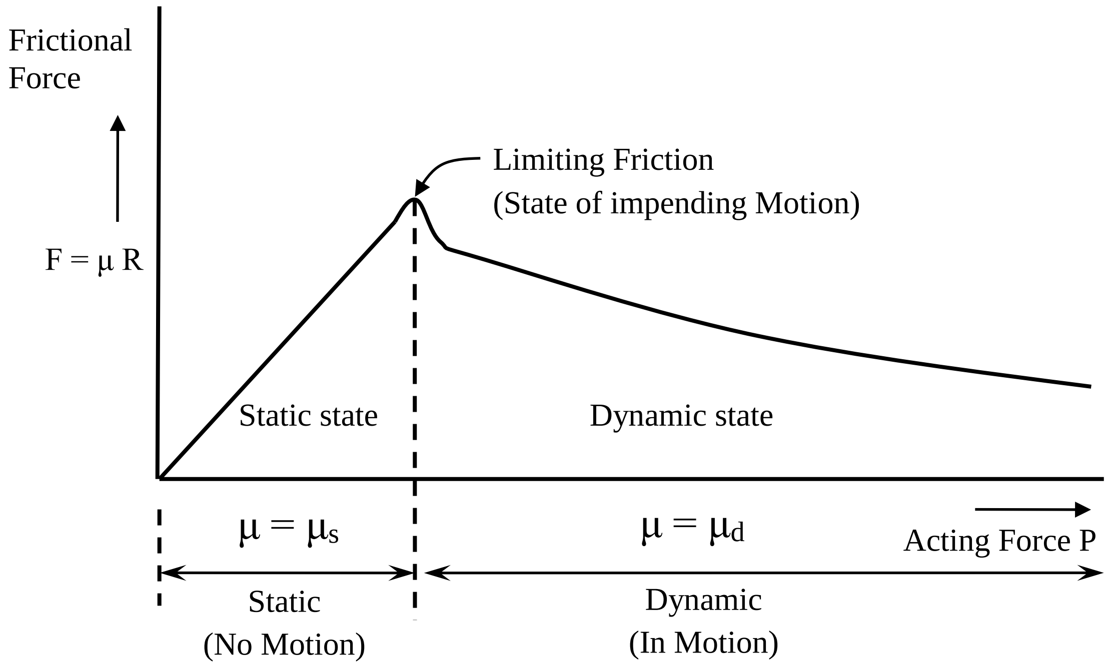
UNIT 2

Whenever the surfaces of two bodies are in contact there will be a limited amount of resistance to sliding between them, which is called friction. The force which opposes the motion or tending motion of one body w. r. t another body is called the "frictional force".

The maximum value of frictional force which makes the motion impending is known as limiting friction.

When the applied force is less than the limiting friction, the body remains at rest and such frictional force is called static friction.

If the applied force exceeds the limiting frictional force, the body starts moving over the other body and the frictional resistance experienced by the body while moving is known as dynamic friction.



Dynamic friction is always less than the static friction.

Sliding friction: it is the friction experienced by the body when it slides over the other body.

Rolling friction: it is the friction experienced by the body when it rolls over the other surface.

Coefficient of friction:

The magnitude of limiting friction bears a constant ratio to the normal reaction between the two surfaces and this ratio is known as coefficient of friction and is denoted by letter 'μ'.

$F \alpha N$

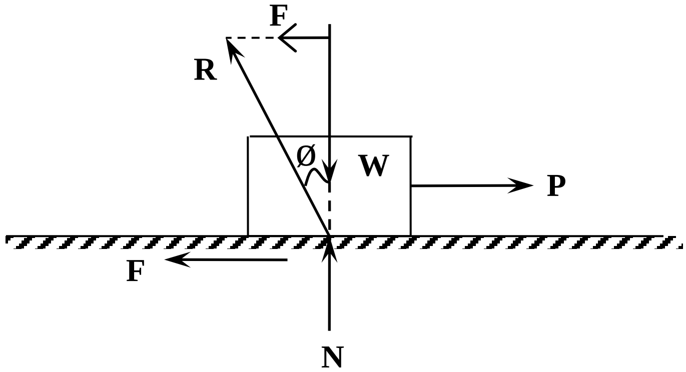
$F = \mu N$

$\mu = \frac{\mathrm{F}}{\mathrm{N}}$ or $\frac{F}{R}$

Let W be the weight of the body, then

Normal reaction N = W

F – Limiting frictional force



P – Applied force

$$F = P, \qquad \mu = \frac{F}{N}$$

Angle of friction:

Let R – resultant of F and N which makes an angle "ø" to the normal.

$$\therefore \ \text{Tan } ø = \frac{F}{N} = \mu$$

As frictional force increases the angle of friction also increases and reaches maximum value of ø when limiting value of friction is reached.

Ø – Angle of limiting friction

Laws of static friction:

1. The maximum force of friction is proportional to the normal reaction on the contact surface.
2. The maximum force of friction is independent of extent of area between two contact surfaces.
3. The frictional force always acts in opposite direction of tending motion of the body.
4. The frictional force depends upon the nature of the surfaces in contact.
5. Limiting static friction is always greater than the dynamic friction.

Angle of repose:

Let a mass of weight 'W' rest on an inclined plane. If the inclination of plane is increased, the magnitude of frictional resistance also increases.

When it reaches the point of maximum frictional resistance, let α be the angle

Resolving normal to the plane, then

$N = W \cos \alpha$ and $F_{max} = \mu N = \mu W \cos \alpha$ ------------- (1)

Let ø be the angle of friction

Resolving W along the plane, then

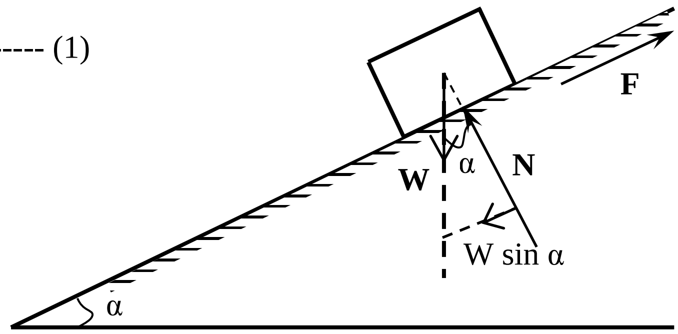
$F_{max} = W \sin \alpha$ ------------- (2)

$\therefore \ \mu W \cos \alpha = W \sin \alpha$

$\mu = \tan \alpha$; but $\mu = \tan ø$

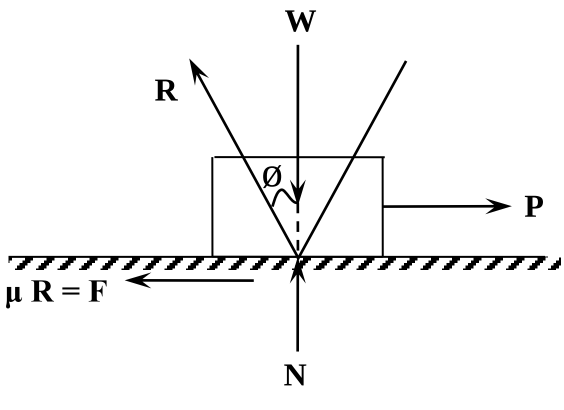
$\text{Tan } \alpha = \tan ø$

$Ø = \alpha$



Cone of static friction:

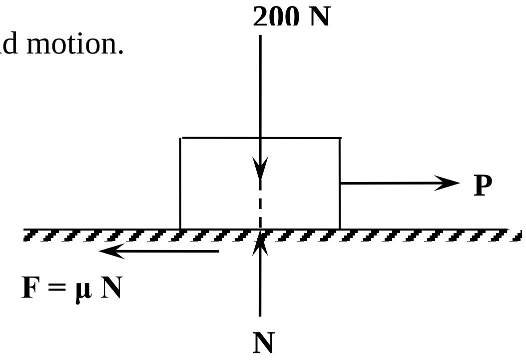
So long as the line of action of the applied force P is completely within a certain cone, generated by a line making the angle of static friction ø with the normal to the surface of contact and having its vertex in this surface, the block will remain in equilibrium regardless of the magnitude of the force. This cone is called the "***cone of static friction***".



Ex: A body of weight 200 N is placed on a rough horizontal plane. If the coefficient of friction between the body and horizontal plane is 0.3, determine

i) Horizontal force required to impend motion.

ii) Pull at an angle 30^0 to horizontal required to impend motion.



Sol: i) given data: N= 200 N, μ = 0.3

$F = \mu N = 200 \times 0.3 = 60$ N.

Resolving horizontally

$P = F = 60$ N.

ii) $F = 0.3$ N

Resolving horizontally

$0.3 N = P \cos 30^0$

$N = 2.887 P$

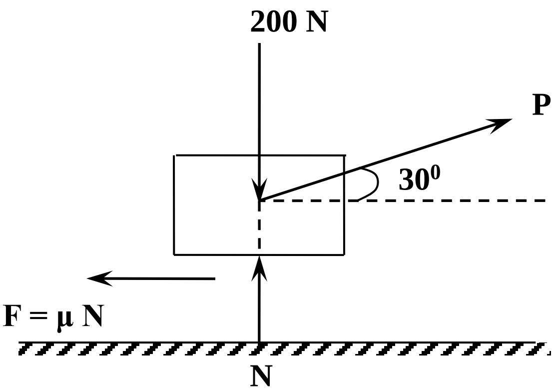
Resolving vertically

$N + P \sin 30^0 = 200$

$2.887 P + 0.5 P = 200$

$3.387 P = 200$

$P = 59.05$ N.



Ex: A body of weight 500 N is pulled up on an inclined plane by a force of 350 N. The inclination of the plane is 30^0 to horizontal and the force is applied parallel to the plane. Determine the coefficient of friction.

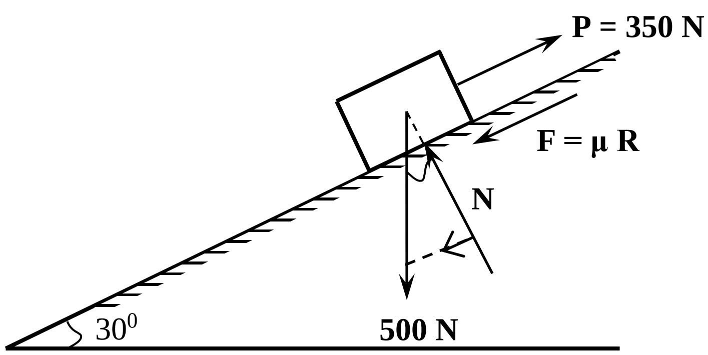
Sol: $N = 500 \cos 30^0 = 433.3$ N

Resolving along the plane,

$P = 500 \sin 30^0 + \mu R$

$350 = 250 + \mu (433.3)$

$\therefore \mu = \frac{100}{433.3} = 0.23.$



Ex: A body of weight 500 N is lying on a rough inclined plane as shown in figure. It is supported by a force P acting parallel to the plane. Determine the max. and min. values of P for which equilibrium can exist. Assume angle of friction as 20^0.

Sol:

i) <u>To obtain P_{max} when it is acting up the plane</u>

Resolving normal to the plane,

$N = W \cos 25^0 = 500 \times 0.963 = 453.15$ N

$\mu = \tan \phi = \tan 20^0 = 0.364$

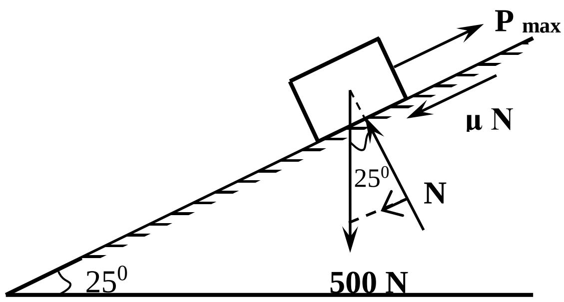
Resolving along the plane

$P_{max} = \mu N + W \sin 25^0$

$= (0.364 \times 453.15) + (500 \times 0.4226)$

$= 164.95 + 211.3$

$P_{max} = 376.25$ N.



Case (i)

ii) <u>To obtain P_{min} when it is acting down / up the plane</u>

Resolving normal to the plane

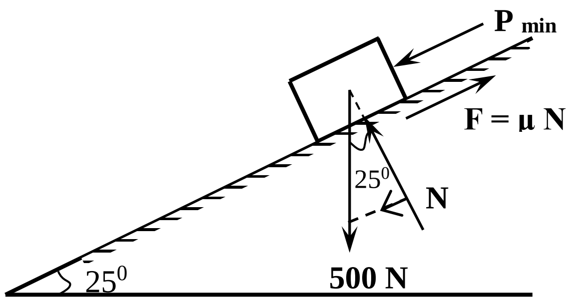
$N = 453.15$ N, $\mu = 0.364$

Resolving along the plane

$P_{min} + W \sin 25^0 = \mu N$

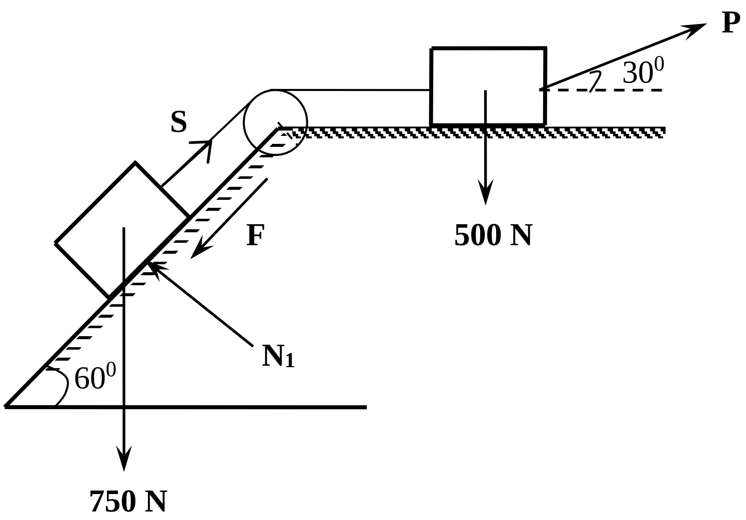
$P_{min} = 164.95 - 211.3 = -46.35$ N.

$\therefore$ P_{min} is also acting upwards to avoid sliding.



Case (ii)

Ex: what is the value of P in the system shown in the figure to cause the motion to impend? Assume the pulley is smooth and coefficient of friction between the other two contact surfaces is 0.20.



Sol: <u>Considering FBD of 750 N</u>

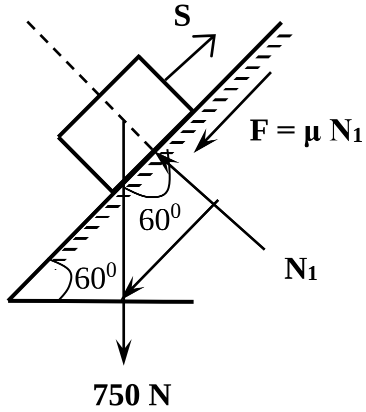
Resolving normal to the plane

$N_1 = 750 \cos 60^0 = 375$ N

Resolving parallel to the plane,

$S = \mu N_1 + 750 \sin 60^0$

$S = (0.2 \times 375) + (750 \times 0.866) = 724.5$ N



<u>Considering FBD of 500 N force</u>

Resolving normal to the plane

$N_2 + P \sin 30^0 = 500$

$N_2 = 500 - 0.5 P$ -------- (1)

Resolving horizontally,

$P \cos 30^0 = 724.5 + 0.2\, N_2$ --------- (2)

Substituting N_2

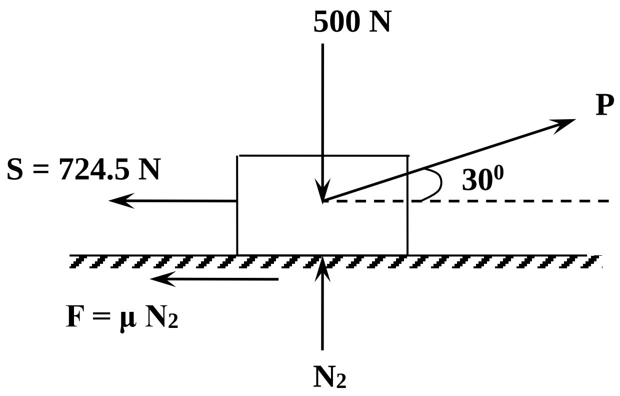
$0.866\, P = 724.5 + 0.2(500 - 0.5\, P)$

$= 724.5 + 100 - 0.1\, P$

$0.966\, P = 824.5$

$P = \frac{824.5}{0.966}$

$P = 853.51$ N.



Ex: A uniform ladder of length 10 m and weighing 20 N is placed against a smooth vertical wall with its lower end 6 m from the wall as shown in the figure. When the ladder is just on the point of slipping, Determine

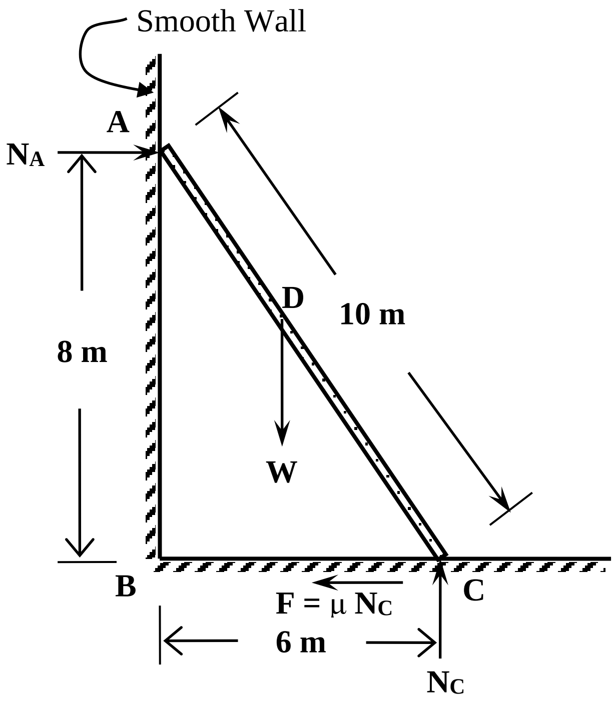
i. Coefficient of friction between ladder and floor
ii. Frictional force between ladder and the floor.

Sol:

$AB = \sqrt{10^2 - 6^2} = \sqrt{100 - 36} = 8$ m.

When the wall is smooth, there will be no frictional force, but normal reaction N_A will be acting perpendicular to the vertical wall.

Let μ be the coefficient of friction between ladder and wall.



Resolving forces vertically

$N_C = W = 20$ N.

Resolving forces horizontally

$N_A = F = \mu N_C = 20\, \mu$.

Taking moments about C,

$N_A . 8 = 20 \times 3$

$20\mu . 8 = 20 \times 3$

$\therefore \mu = 0.375$

Frictional force $F = \mu N_C = 0.375 \times 20 = 7.5$ N.

Ex: What should be the value of 'θ' in figure, which will make the motion of 900N block down the plane to impend? The coefficient of friction for all contact surfaces is 1/3.

Sol: <u>Considering F B D of 300N force</u>

When 900 N block impends down, the relative motion of 300 N block is upwards. Hence frictional force acts downwards.

Resolving forces normal to the plane

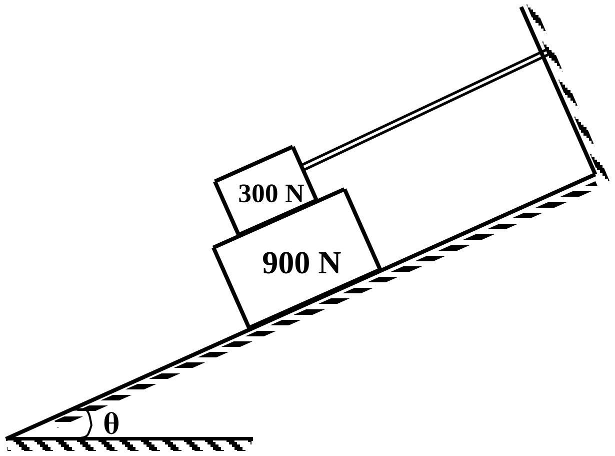
$N_1 = 300 \text{ Cos } \theta$

Resolving forces along the plane

$S = \mu N_1 + 300 \text{ Sin } \theta$

$S = \frac{1}{3} \times 300 \text{ Cos } \theta + 300 \text{ Sin } \theta$

$S = 100 (\text{Cos } \theta + 3 \text{ Sin } \theta)$



Considering F B D of 900N force

Resolving forces normal to the plane

$N_2 = 900 \text{ Cos } \theta + 300 \text{ Cos } \theta$

$N_2 = 1200 \text{ Cos } \theta$

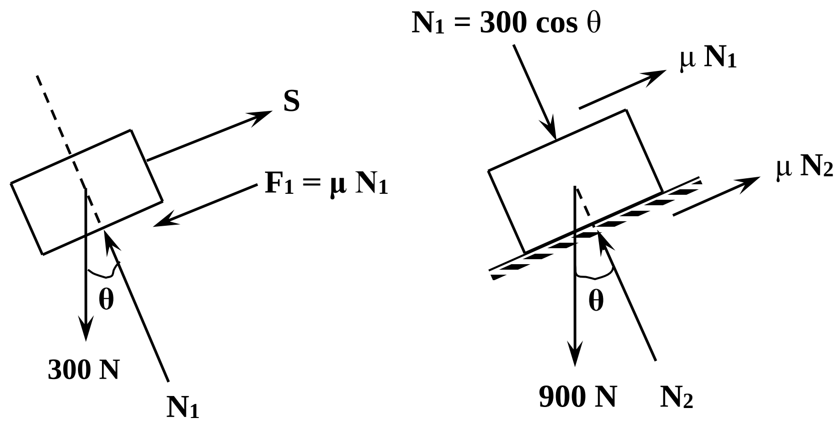
Resolving forces along the plane

$\mu (N_1 + N_2) = 900 \text{ Sin } \theta$

$\frac{1}{3} (1500 \text{ Cos } \theta) = 900 \text{ Sin } \theta$

$\therefore \text{ Tan } \theta = \frac{500}{900} = \frac{5}{9}$

$\theta = \text{Tan}^{-1}(\frac{5}{9}) = 29^0.05$



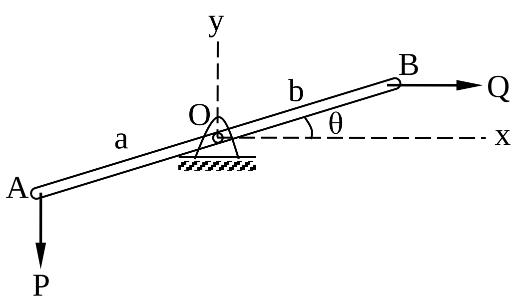
Virtual work:

In previous chapters, equilibrium of rigid bodies or connected rigid bodies by various support conditions subjected to applied forces are only discussed.

But calculation of reactions can be obtained for a system defining the possible configurations of equilibrium when it has some freedom of motion.

Virtual displacement:

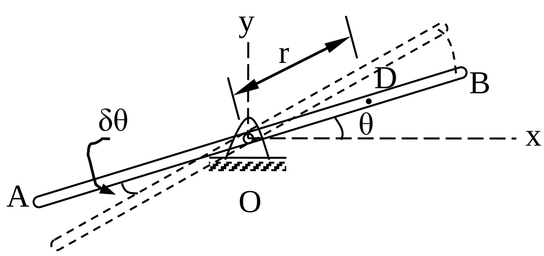
Let AB be the lever hinged at 'O' and it is free to rotate about z-z axis i.e., normal to the plane of the figure and acted upon by forces P & Q acting in the same plane. Then the position of the lever is completely defined by the angle 'θ' that makes with x – axis. Then 'θ' can be obtained if P & Q forces are specified. Since the rotation is only about z – z axis, it is single degree of freedom.



θ defining the position of lever is called " coordinate" of the system.

Let θ is given a small increment δθ at any point D on the axis of lever at a distance 'r' from O will describe an infinitesimal arc r.dθ which is perpendicular to AB.

Such displacements are called "Virtual Displacements".

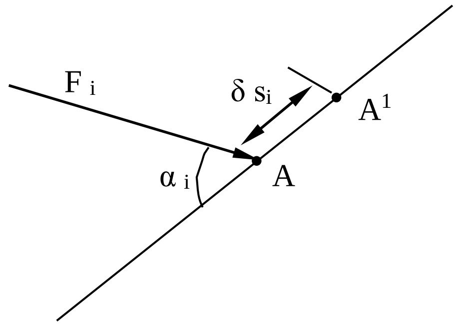


Ex: virtual displacements at A = a δθ, at B = b δθ

Virtual work: (Virtual means imaginary)

Consider a force F_i acts at a point A and virtually displaces by δs_i from A to A[1] as shown in figure below.

Then virtual work done by F_i as the product of force component and virtual displacement δs_i



Therefore $\delta v_i = (F_i \cos \alpha_i)\, \delta s_i$

Where $F_i \cos \alpha_i$ is force component in the direction of displacement

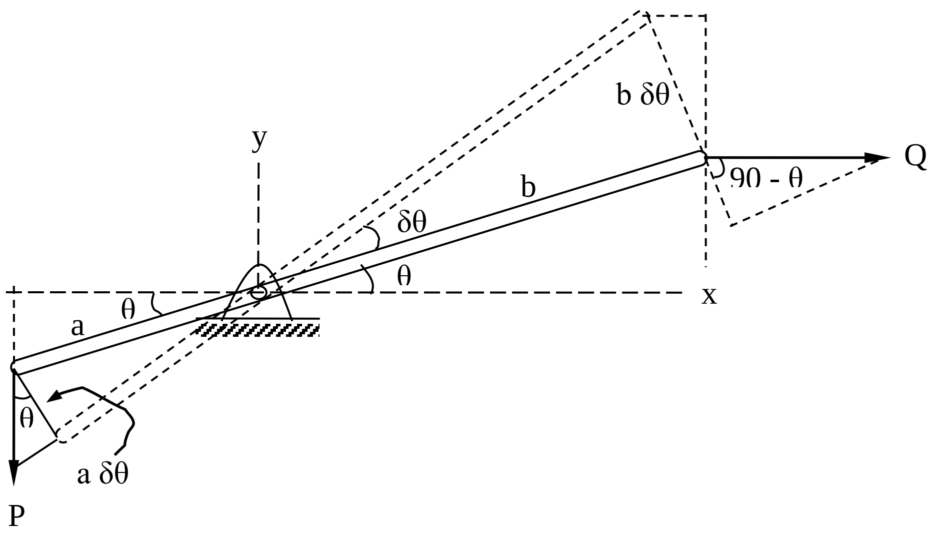
This work done is considered positive when displacement is in the direction of force component and negative when the displacement is in opposite direction of force component.

While dealing virtual displacements & work done, it is assumed that there is no friction in hinges, bearings or along sliding surfaces such that the parts of system behave rigid. Such systems are called "Ideal Systems" i.e., only the applied forces will do virtual work on virtual displacements of ideal system.

Statement of virtual work:

If for each displacement of an ideal system, the work produced by the active forces is zero, then the system is in a configuration of equilibrium.

$\Sigma F_i \cos \alpha_i \, \delta s_i = 0$ for all active forces



Applying to an example of lever

$$+ (P \cos \theta)\, a\, \delta\theta - Q \cos (90 - \theta).\, b\, \delta\theta = 0$$

$$+ P \cos\theta \;\; a\, \delta\theta - Q \sin\theta.\, b.\, \delta\theta = 0$$

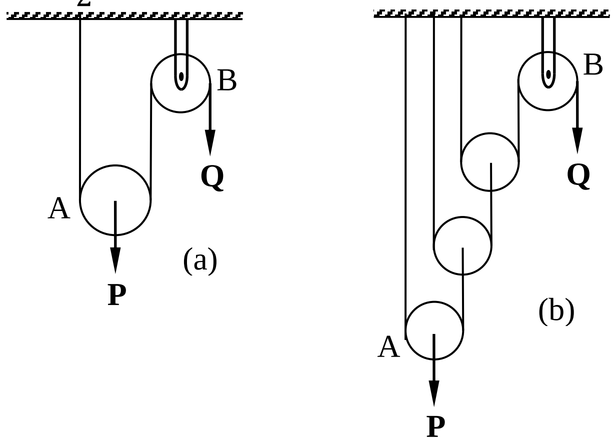
$$\tan \theta = \frac{P\, a}{Q\, b}$$

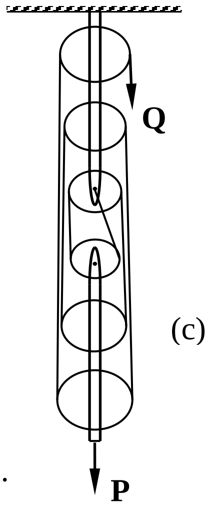
<u>Considering pulley an example of system:</u>

(a) If Q moves δx downwards P moves up by $\frac{\delta x}{2}$.

$$Q.\delta x - P.\frac{\delta x}{2} = 0$$

$$Q = \frac{P}{2}$$





(b) Upward displacement at A is $\frac{1}{8}$ of the downward displacement of B (Q).

$$Q.\delta x - P.\frac{1}{8}\delta x = 0$$

$$Q = \frac{P}{8}$$

(c) For the given configuration of pulley systems,

$$Q = \frac{P}{6}$$

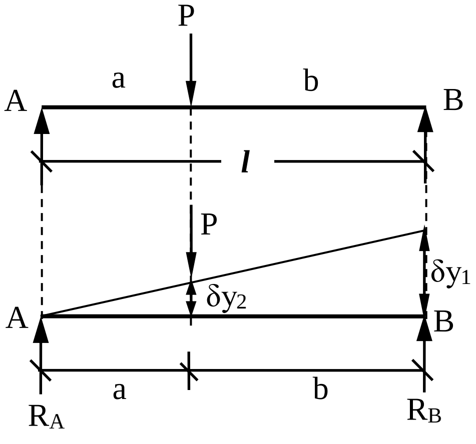
<u>Note:</u> Virtual work becomes zero when

1) Tensions produced in an inextensible string
2) Reactions at smooth pins and hinges
3) Reactions on rollers
4) Forces normal to the direction of virtual displacements
5) Mutual action and reaction between two bodies.

Important Points:

1. If the tension 'T' in an inextensible string is to be calculated, replace the string by two forces each of 'T' acting at the ends. Now apply principle of virtual work to calculate 'T'.
2. If 'R' reaction at support is to be calculated, remove the support and replace it by a force 'R'.
3. If all the forces are acting in the same direction, then transfer the direction of virtual displacements in the direction of forces.

<u>Example of beam:</u>



Let virtual displacement δy_1 is given at B.

Virtual work done by $R_B = + R_B.\ \delta y_1$

Virtual work done by $R_A = + R_A \times 0 = 0$.

Virtual work done by $P = - P.\ \delta y_2$

Total work done $\quad U = + R_B.\ \delta y_1 + 0 - P.\ \delta y_2 = 0$

$$R_B.\ \delta y_1 = P.\ \delta y_2 \qquad \text{--- (1)}$$

But $\dfrac{\delta y_1}{\delta y_2} = \dfrac{a+b}{a} = \dfrac{l}{a}$

$\delta y_1 = \dfrac{l}{a}\delta y_2 \quad --- (2)$

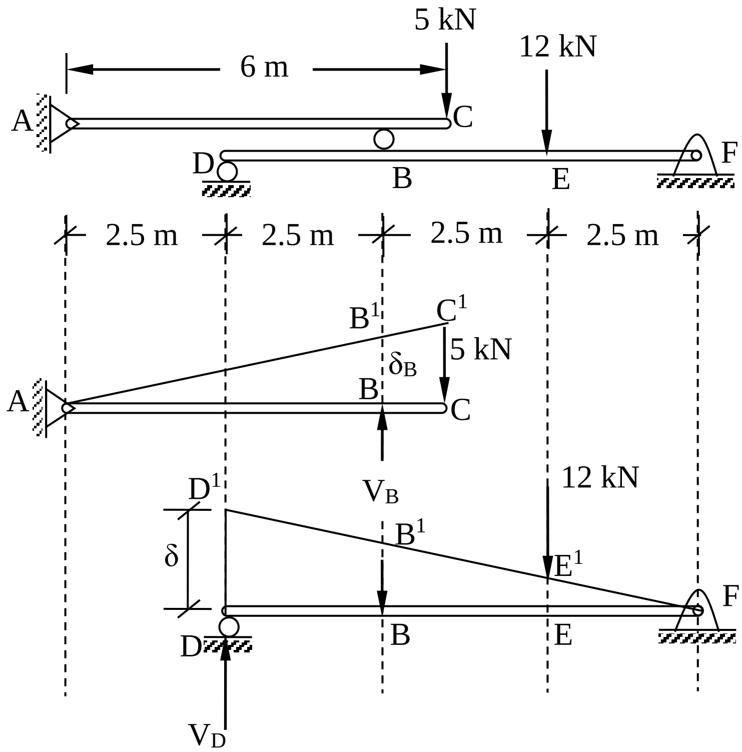
Substituting in (1)

$R_B.\dfrac{l}{a}\delta y_2 = P.\delta y_2$

$\therefore R_B = \dfrac{P\,a}{l}$

similarly $R_A = P - \dfrac{P\,a}{l} = \dfrac{P\,(a+b) - P\,a}{l} = \dfrac{P\,b}{l}$

$\therefore R_A = \dfrac{P\,b}{l}$



Ex: find the reactions at rollers B & D for the loaded beam shown in fig.

Sol: Consider beam AC

Let δ_B – displacement of reactions V_B

$CC^1 = \delta_B \frac{6}{5}$

Total virtual work done

$= V_B.\delta_B - 5 \times \dfrac{6}{5}\,\delta_B = 0$

$V_B = 6$ kN.

Consider beam DF

Let displacement of reaction $V_D = \delta_D$

$BB^I = \dfrac{5}{7.5}\,\delta_D = \dfrac{2}{3}\,\delta_D$

$EE^I = \dfrac{2.5}{7.5}\,\delta_D = \dfrac{1}{3}\,\delta_D$

Total virtual work done

$= V_D \times \delta - V_B.\,BB^l - 12\,EE^1 = 0$

$V_D.\delta - 6 \times \dfrac{2}{3}\,\delta_D - 12 \times \dfrac{1}{3}\,\delta_D = 0$

$V_D = 4 + 4 = 8$ kN.

Ex: A beam supported on two identical rollers each of radius 'r', is moved up an inclined plane by the application of a force 'P' acting parallel to the plane as shown in figure below. If the weight of the beam is 'Q' and the weight of each roller is 'W', using virtual work principle, find the force 'P' applied for equilibrium. Assume no slip between the rollers and the inclined plane of between rollers and the beam.

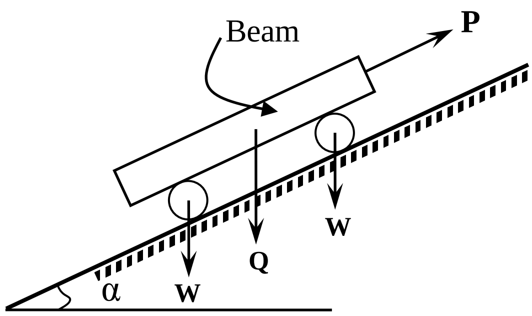
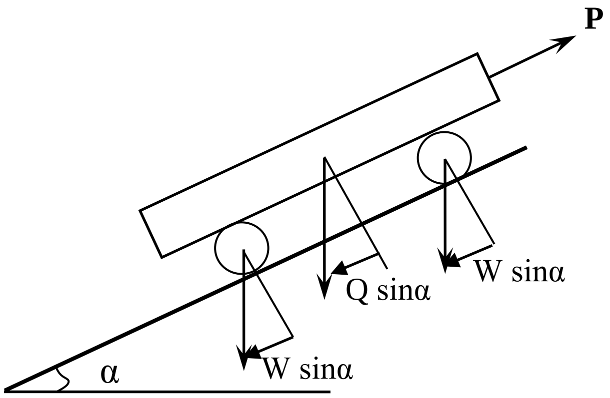


Fig.



Sol:

Since there is no slipping, let each roller move up by displacement δx.

Displacement of beam = 2 δx up the plane

Applying virtual work principle

$+ P.\ 2\ \delta x - 2\ W \sin\alpha.\ \delta x - Q \sin\alpha.\ 2.\ \delta x = 0$

$P = (W + Q) \sin\alpha$

UNIT –III

Objective:

To understand the basic concepts of centroid, center of gravity and moment of inertia and apply to simple examples of plane figures and bodies.

Syllabus:

Centroid & Centre of gravity: Definition of centroid and centre of gravity - Mathematical expressions for centroid and centre of gravity, use of symmetrical axis. Centroids of standard figures (from first principles) like Triangle, Sector of a circle, Semi circle, Quadrant of a circle, Quadrant of an ellipse - Centroid of composite figures.

Centre of gravity of standard bodies (from first principles) like Cone and Hemisphere – Centre of gravity of composite bodies - Pappu’s and Guldinus theorems.

Area Moment of Inertia:

Definition of Moment of inertia, Mathematical expression for Moment of Inertia - Polar moment of inertia – Transfer or Parallel axis theorem – Moment of inertia of standard figures (from first principles) like rectangle, triangle, circle, semi-circle –Moment of inertia of composite figures – Radius of gyration.

Learning Outcomes:

Student will be able to

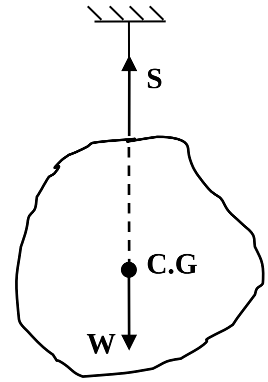
- Distinguish between centroid and center of gravity
- Obtain the mathematical expressions for centroid and center of gravity of areas and bodies
- Derive from first principles, the centroid and center of gravity of standard figures and bodies respectively
- Calculate centroids of composite figures
- Obtain the mathematical expressions for area moment of inertia about x and y axis
- Apply the parallel axis theorem and perpendicular axis theorem to composite sections.
- Derive expressions for moment of inertia of standard figures
- Calculate the moment of inertia of composite figures about both x and y axis passing through its centroid
- Obtain radius of gyration of standard figures.

Centroid and Center of gravity:

Centroid, center of gravity and moment of inertia are the important properties of areas or bodies which are necessary for analysis of any engineering problems. When the shape of the body is responsible for its behavior, these parameters are important for engineering application, analysis and design.

Ex: C.G of a boat, to determine the eccentricity, design of beams, girders etc.

When the line of tensile force in string (S) and the line of weight of body passing through the C.G point of the body is in collinear then system is in equilibrium.



Centroid of an irregular plane figure:

Let δA_1, δA_2, δA_3….. are the element areas x_1, x_2, x_3….. are the distances of elements from y axis and y_1, y_2, y_3….. are the distances from x – axis.

$\delta A_1 + \delta A_2 + \delta A_3 \ldots = \sum_{i=1}^{n} \delta A_i = A$.

Using integral principle,

$$A = \int_{i=1}^{n} dA_i$$

Taking moments of all element areas about x-axis,

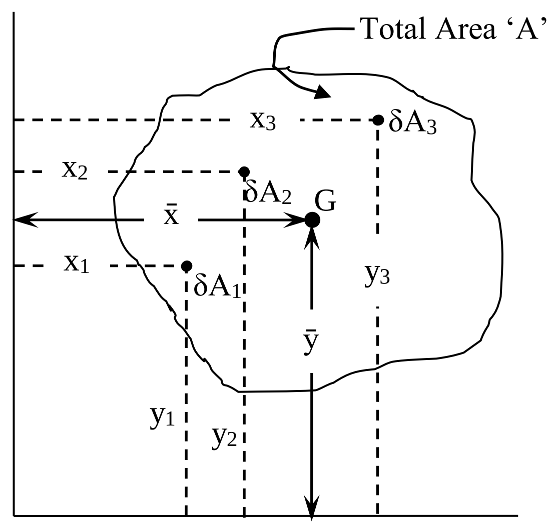
$\delta A_1.y_1 + \delta A_2.y_2 + \delta A_3.y_3 \ldots = \sum_{i=1}^{n} \delta A_i.y_i$

Using integral principle,

$A.\bar{y} = \int_{i=1}^{n} y_i.\delta A_i$ (but $A = \int dA$)

$$\therefore \bar{y} = \frac{\int y\, dA}{\int dA} \quad \ldots (1)$$

Similarly $\bar{x} = \frac{\int x\, dA}{\int dA}$ …. (2)



Centroid is a point in a plane area such that the moment of area about any axis passing through that point is zero. **Centroid** is a point at which the total area of the figure is assumed to be concentrated and is represented by a letter 'G' or 'C'.

Centroid of irregular shaped lines:

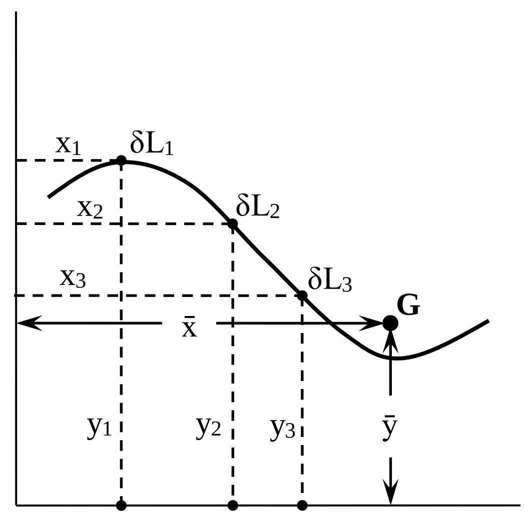
Total length = L = $\int dL$

Σ moments of individual element lengths about x axis = $\int y\, dL$

Σ moments of individual element lengths about y axis = $\int x\, dL$

$\therefore L.\bar{y} = \int y\, dL$ $\qquad$ $\therefore L.\bar{x} = \int x\, dL$

$\bar{y} = \frac{\int y\, dL}{L} = \frac{\int y\, dL}{\int dL}$ $\qquad$ $\bar{x} = \frac{\int x\, dL}{L} = \frac{\int x\, dL}{\int dL}$



Centre of gravity: It is defined as a point through which the resultant force of gravity or weight of the body appears to be concentrated for any position of the body.

Element weight = $\delta W_1 = \rho\, \delta V_1$

$\delta W_2 = \rho\, \delta V_2$

$\delta W_3 = \rho\, \delta V_3$

Total weight $W = \int dW = \rho \int dV$

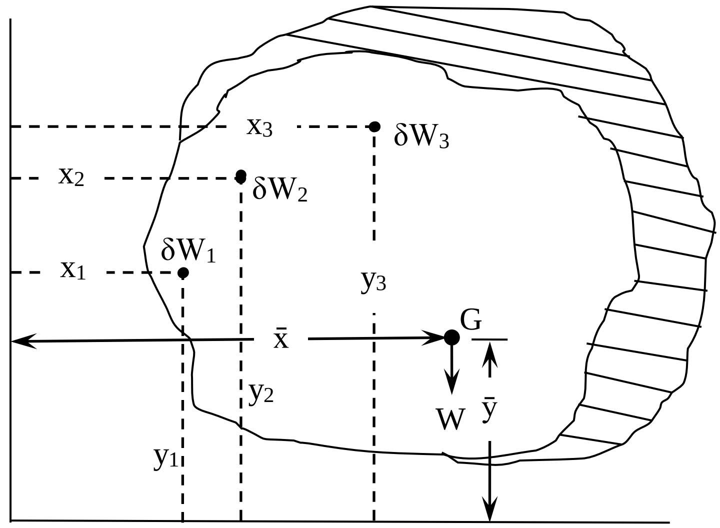
Sum of moments of all element weights about x axis

$= \Sigma\, dW_i \, .y_i$

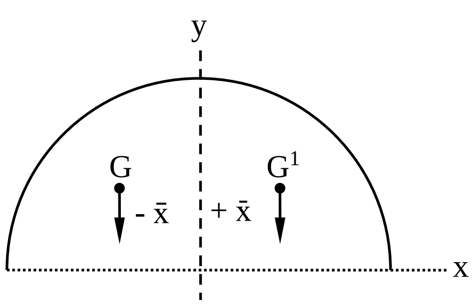
In integral terms, $W\,\bar{y} = \int y_i \, dW_i$

$\bar{y}\, \rho \int dV = \rho \int y_i \, dV$

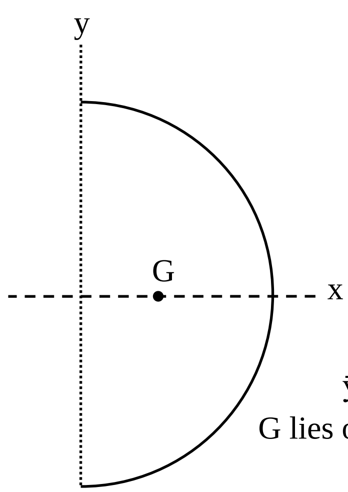
$$\therefore \bar{y} = \frac{\int y\, dV}{\int dV}$$



Use of axes of symmetry while calculating centroids:



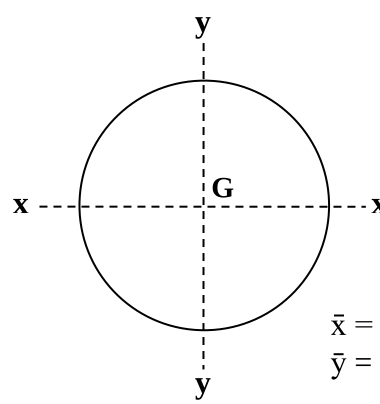
Semi Circle



$\bar{y} = 0$

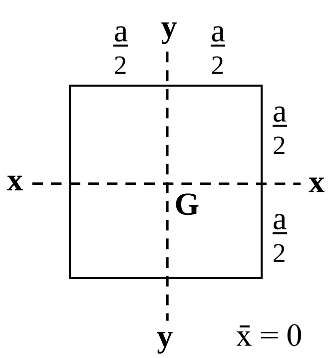
G lies on x - axis

Semi Circle



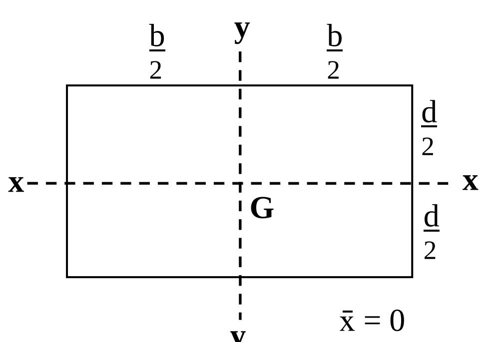
$\bar{x} = 0$

$\bar{y} = 0$



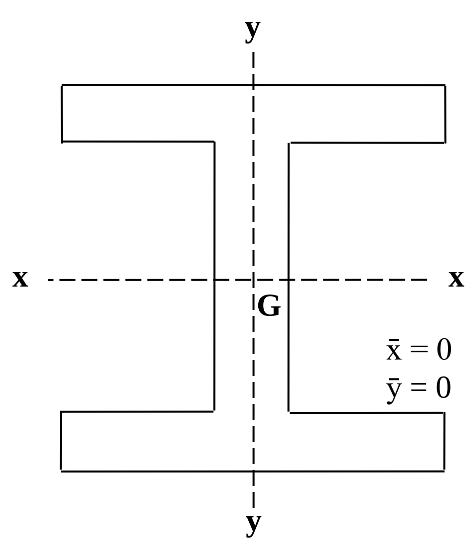
$\bar{x} = 0$

$\bar{y} = 0$



$\bar{x} = 0$

$\bar{y} = 0$



$\bar{x} = 0$

$\bar{y} = 0$

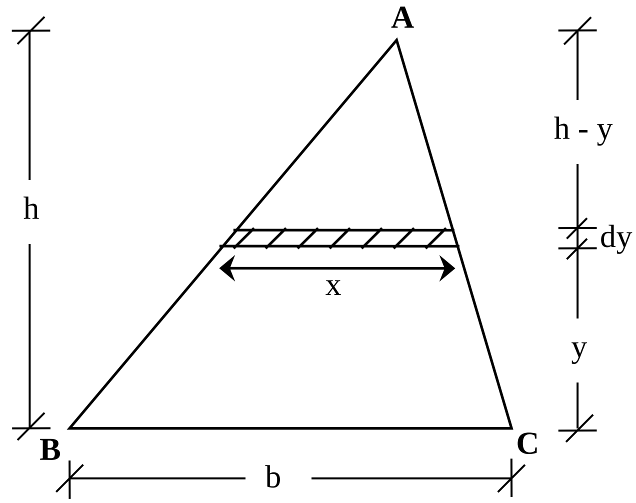
Centroid of a triangle whose base width is 'b' and altitude height 'h':

$$\bar{y} = \frac{\int y\,dA}{\int dA}$$

$$\frac{x}{b} = \frac{h-y}{h}$$

$$\therefore x = \frac{b}{h}(h-y)$$

$$dA = x.dy = \frac{b}{h}(h-y)dy$$



$$A = \int dA = \int_0^h \frac{b}{h}(h-y)dy = \frac{b}{h}\left[hy - \frac{y^2}{2}\right]_0^h$$

$$A = \frac{b}{h}\left[h^2 - \frac{h^2}{2}\right] = bh - \frac{bh}{2} = \frac{bh}{2}$$

$$\int y\,dA = \int_0^h \frac{b}{h}(h-y)y.dy = \frac{b}{h}\int_0^h (hy - y^2)\,dy$$

$$= \frac{b}{h}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h = \frac{b}{h}\left[\frac{h^3}{2} - \frac{h^3}{3}\right] = \frac{bh^2}{6}$$

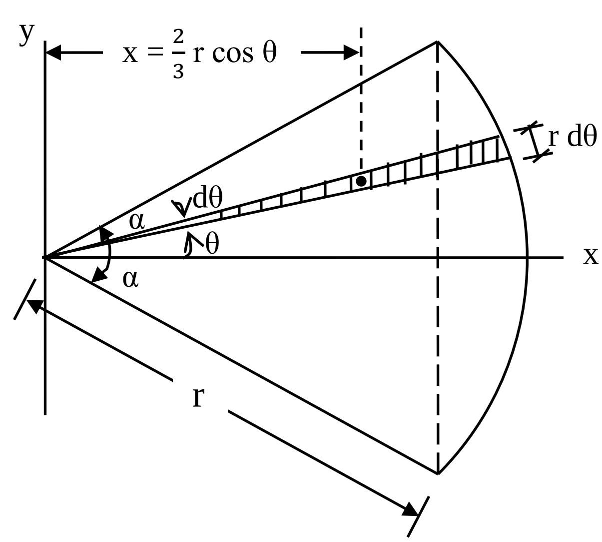
$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{bh^2}{6 \times \frac{bh}{2}} = \frac{h}{3}$ from base or $\frac{2}{3}h$ from top.

Centroid of area of a circular sector:

x – x is axis of symmetry

$\therefore \bar{y} = 0$

Let an element sector is considered as shown



$$dA = \frac{1}{2}r.r\,d\theta = \frac{r^2}{2}\,d\theta$$

$$A = \int dA = \int_{-\alpha}^{+\alpha} \frac{r^2}{2}\,d\theta$$

$A = \frac{r^2}{2}[\alpha - (-\alpha)] = r^2\,\alpha$

$$\bar{x} = \frac{\int x\,dA}{\int dA}$$

$$\int x\,dA = \int \frac{2\,r}{3}\,cos\theta\,.\frac{r^2}{2}.d\theta = \frac{r^3}{3}\int_{-\alpha}^{+\alpha}\cos\theta\,d\theta = \frac{r^3}{3}\,[2\sin\alpha]$$

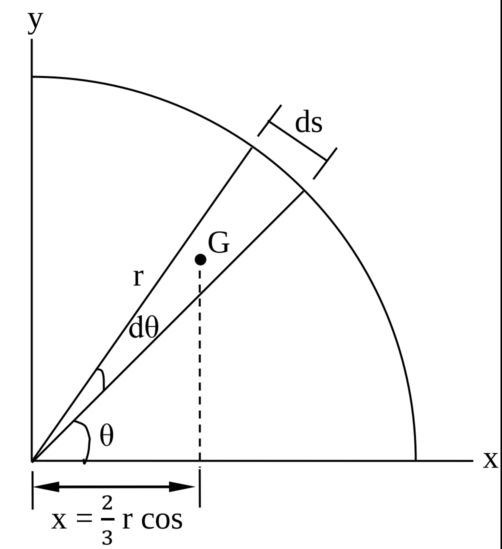
$$\therefore \bar{x} = \frac{\int x\,dA}{\int dA} = \frac{r^3}{3}\,\frac{2\sin\alpha}{r^2\,\alpha} = \frac{2}{3}\,\frac{r\sin\alpha}{\alpha}$$

In case of semi circle, $\alpha = 90^0$

$$\bar{x} = \frac{2}{3}\,\frac{r\sin\alpha}{\alpha} \quad = \quad \frac{2}{3}\,r\;\frac{1}{\frac{\pi}{2}}$$

$$\bar{x} = \frac{4r}{3\pi} = 0.424\ r$$

Centroid of quadrant of a circle:



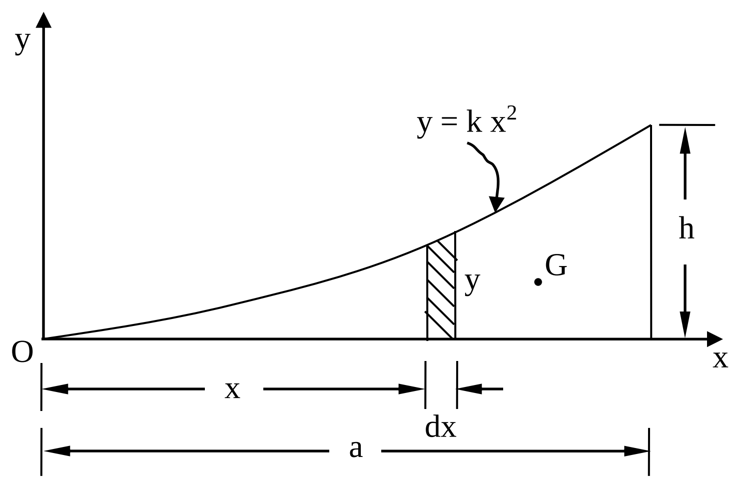
$$dA = \frac{1}{2}\,r.r\,d\theta = \frac{r^2}{2}.d\theta$$

$$A = \int dA = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}.d\theta = \frac{r^2}{2}\,[\,\theta\,]_0^{\frac{\pi}{2}} = \frac{\pi\,r^2}{2\times 2} = \frac{\pi\,r^2}{4}$$

$$\int x.dA = \int_0^{\frac{\pi}{2}} \frac{2}{3}\,r.\cos\theta\,.\frac{r^2}{2}.d\theta = \frac{r^3}{3}\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta = \frac{r^3}{3}\,[\sin\theta\,]_0^{\frac{\pi}{2}} = \frac{r^3}{3}$$

$$\therefore\ \bar{x} = \frac{\int x\,dA}{\int dA} = \frac{r^3}{3}\,\frac{1}{\frac{\pi\,r^2}{4}} = \boldsymbol{\frac{4}{3}\,\frac{r}{\pi}}$$

Centroid of a parabolic spandrel:



Equation of parabola $y = kx^2$

Area of element $dA = y\,dx = kx^2\,dx$

Total area $A = \int_0^a kx^2\,dx = k\left[\frac{x^3}{3}\right]_0^a = \frac{ka^3}{3}$

Moment of dA about y – axis

$$= \int x\,dA = \int_0^a k\,x^3\,dx = k\left[\frac{x^4}{4}\right]_0^a = \frac{k\,a^4}{4}$$

Moment of dA about x – axis

$$\int \frac{y}{2}\,dA = \int_0^a \left(\frac{k\,x^2}{2}\right) k\,x^2\,dx = \int_0^a \left(\frac{k^2x^4}{2}\right) dx = \frac{k^2}{2}\left[\frac{x^5}{5}\right]_0^a = \frac{k^2a^5}{10}$$

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{k\,a^4}{4\times\frac{k\,a^3}{3}} = \frac{3}{4}a$$

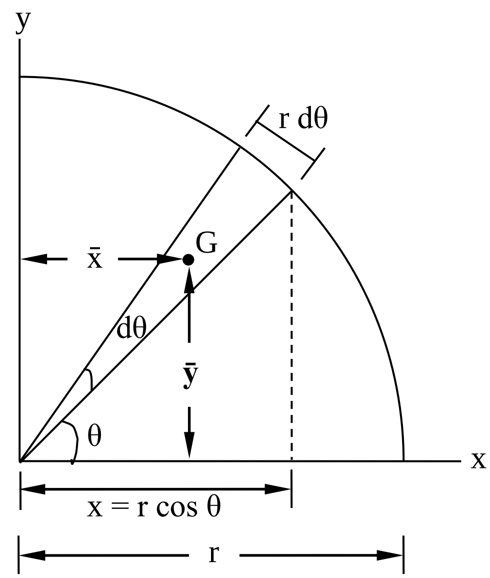
$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{k^2a^5}{10\times\frac{k\,a^3}{3}} = \frac{3}{10}\,k\,a^2$$

At $x = a$, $y = h$ in the equation $y = k\,x^2$

$\therefore\ h = k\,a^2$; $k = \frac{h}{a^2}$; $\bar{y} = \frac{3}{10}.\frac{h}{a^2}.a^2$

$\therefore \bar{y} = \frac{3}{10}.h \qquad ; \bar{x} = \frac{3}{4}a$

Centroid of the arc of a quadrant of a circle:



Take an element length dL, $dL = r\,d\theta$

$x = r\cos\theta$

$$\bar{x} = \frac{\int x\,dL}{\int dL}$$

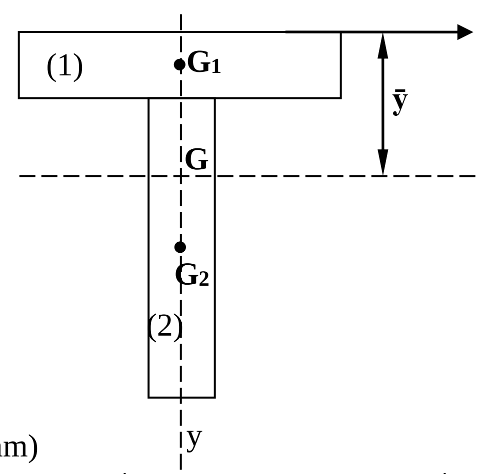
$$\int dL = \int_0^{\frac{\pi}{2}} r\,d\theta = r.\frac{\pi}{2} = \frac{\pi r}{2}$$

$$\int x\,dL = \int_0^{\frac{\pi}{2}} r\cos\theta\,.r\,d\theta = r^2\int_0^{\frac{\pi}{2}} \cos\theta\ d\theta = r^2[\sin\theta]_0^{\frac{\pi}{2}} = r^2$$

$$\bar{x} = \frac{\int x\,dL}{\int dL} = \frac{r^2}{\frac{\pi r}{2}} = \frac{2r}{\pi}$$

Centroids of composite areas:

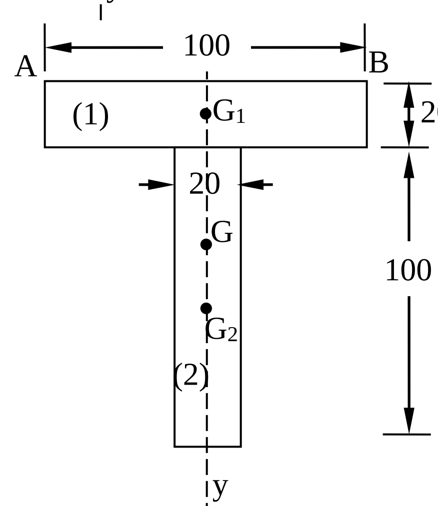
Divide the figure into known standard figures and apply the same principle. Sum of moments of individual areas about an axis is equal to the sum of the areas of the figure multiplied by its centroidal distance from that axis.



$(A_1 + A_2)\,\bar{y} = A_1 y_1 + A_2 y_2$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$$

Ex: Locate the centroid of T – section shown in figure. (Units are in mm)



Sol: Since there is symmetry about y axis, $\bar{x} = 0$

To calculate $\bar{y}$ from top, taking moments about AB,

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$$

$$\bar{y} = \frac{(100 \times 20)10 + (100 \times 20)(20 + 50)}{(100 \times 20) + (100 \times 20)}$$

$$\bar{y} = \frac{20000 + 140000}{4000} = \frac{160}{4} = 40\ mm$$

$\therefore\ \bar{y} = 40$ mm from the top.

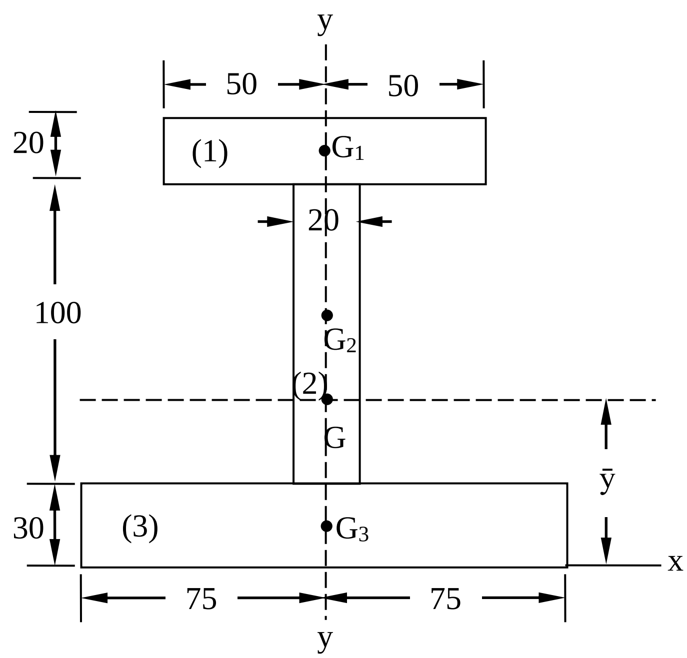
Ex: Locate the centroid of the following figure.

Sol: $A_1 = 100 \times 20 = 2000$

$A_2 = 100 \times 20 = 2000$

$A_3 = 150 \times 30 = 4500$

$y_1 = 140,\ y_2 = 80,\ y_3 = 15$



Since there is symmetry about y axis, $\bar{x} = 0$

Taking moments about the base

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3}$$

$$\bar{y} = \frac{(2000 \times 140) + (2000 \times 80) + (4500 \times 15)}{2000 + 2000 + 4500}$$

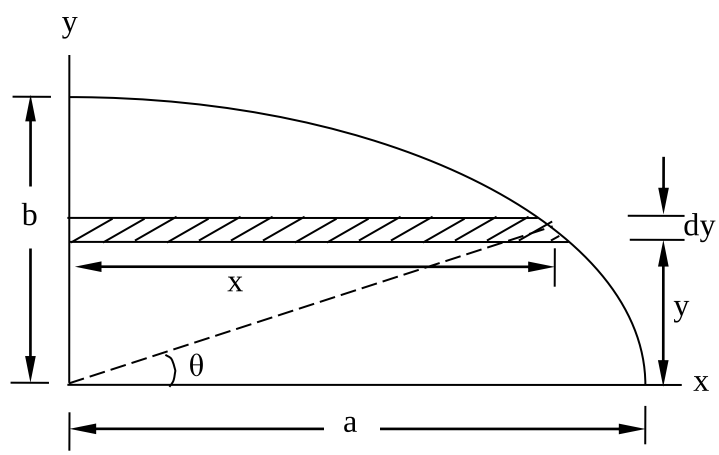
$$\bar{y} = \frac{2800 + 1600 + 675}{85} = \frac{5075}{85} = 59.71\ mm$$

$\therefore\ \bar{x} = 0$

$\bar{y} = 59.71$ mm from x – axis.

Ex: Determine the centroid of quadrant of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol: Area of element dA = x. dy



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 = a^2 \left[\frac{b^2 - y^2}{b^2}\right]\ and\ x = \frac{a}{b}\sqrt{b^2 - a^2}$$

Let $x = a \cos\theta,\ y = b \sin\theta$

$dy = b \cos\theta\, d\theta$

When $y = 0,\ \theta = 0$

When $y = b,\ \theta = 90^0$

<u>To determine $\bar{y}$,</u>

$$\bar{y} = \frac{\int y\, dA}{\int dA}$$

$$\int dA = \int_0^b \frac{a}{b}\sqrt{b^2 - y^2}\, dy = \int_0^{\frac{\pi}{2}} \frac{a}{b}\left(\sqrt{b^2 - b^2 sin^2\theta}\right) b\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{a\,b}{b}\left(\sqrt{1 - sin^2\theta}\right) b\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} \frac{ab^2}{b}\cos^2\theta\, d\theta = ab\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = ab\,.\frac{1}{2}.\frac{\pi}{2} = \boldsymbol{\frac{\pi\, a\, b}{4}}$$

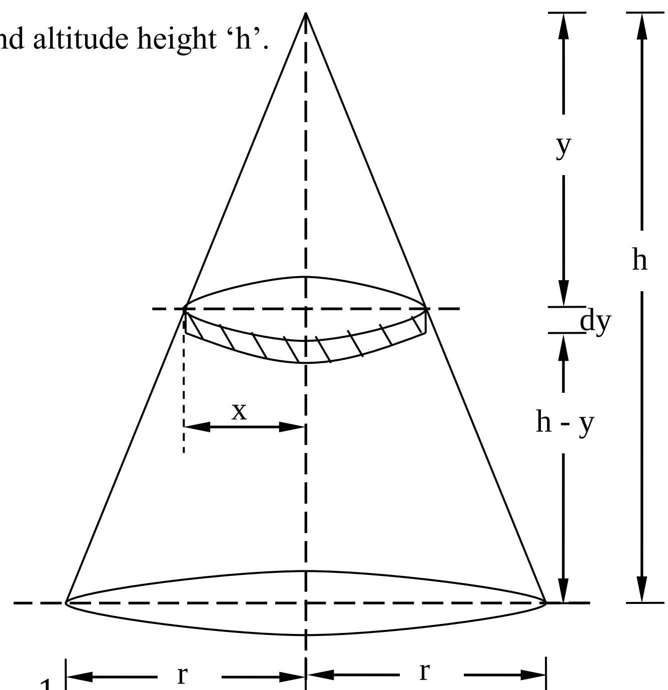
$$\int y\,dA = \int_0^{\frac{\pi}{2}} b\sin\theta\ \frac{a}{b}\ (\sqrt{b^2 - b^2 sin^2\theta})\ b\cos\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} ab^2 \sin\theta\ cos^2\theta\ d\theta = \left[ab^2.\frac{cos^3\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$\int y\,dA = \frac{ab^2}{3}$$

$\therefore \bar{y} = \frac{\int y\,dA}{\int dA} = \frac{ab^2}{3\times\frac{\pi a b}{4}} = \frac{4\,b}{3\,\pi}$ Similarly $\bar{x} = \frac{4\,a}{3\,\pi}$

Ex: Find the C.G of a right circular cone of base radius 'r' and altitude height 'h'.



Sol: $dV = \pi\, x^2.\, Dy$

$$\frac{x}{r} = \frac{y}{h}$$

$\therefore x = \frac{r}{h}.y$ $\therefore dV = \frac{\pi r^2.y^2}{h^2}dy$

$$\int dV = \int_0^h \frac{\pi r^2}{h^2}.y^2\,dy$$

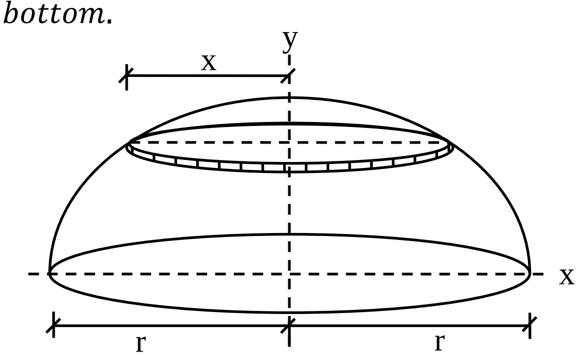
$$\therefore V = \int dV = \frac{\pi r^2}{h^2}.\left[\frac{y^3}{3}\right]_0^h = \frac{\pi r^2 h^3}{3h^2} = \frac{1}{3}\pi\, r^2 h$$

$$\therefore \bar{y} = \frac{\int y\,dV}{\int dV}$$

$$\int y\,dV = \int_0^h y.\frac{\pi r^2}{h^2}.y^2\,dy = \int_0^h \frac{\pi r^2}{h^2}.y^3\,dy = \frac{\pi r^2}{h^2}.\left[\frac{y^4}{4}\right]_0^h = \frac{1}{4}\pi\, r^2 h^2$$

$\therefore \bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\pi\, r^2 h^2}{4\times\frac{1}{3}\pi\, r^2 h} = \frac{3}{4}$h from top $or\ \bar{y} = \frac{1}{4}h\ from\ bottom.$

Ex: Determine the centroid of a hemi sphere whose radius is 'r'.



Sol: take an element strip of thickness 'dy' and radius 'x' at a distance 'y' from x – axis.

Volume of strip $dV = \pi\, x^2\, .dy$

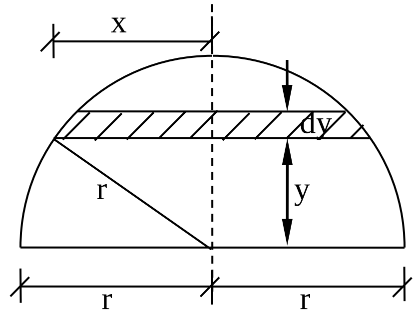
$$x^2 + y^2 = r^2$$

$$x^2 = r^2 - y^2$$

$$dV = \pi\,(r^2 - y^2)\,dy$$

$$\int dV = \int_0^r \pi\,(r^2 - y^2)dy = \pi\left[r^2 y - \frac{y^3}{3}\right]_0^r$$

$V = \int dV = \pi \left[r^3 - \frac{r^3}{3} \right] = \frac{2}{3} \pi r^3$

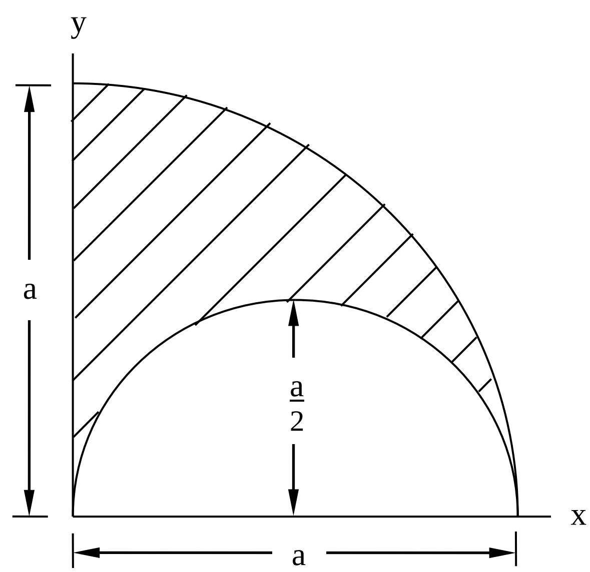


$$\int y\, dV = \int_0^r \pi\,(r^2 - y^2) y\, dy = \pi \int_0^r (r^2 y - y^3) dy$$

$$\int y\, dV = \pi \left[\frac{r^2 y^2}{2} - \frac{y^4}{4} \right]_0^r = \pi \left[\frac{r^4}{2} - \frac{r^4}{4} \right] = \frac{\pi r^4}{4}$$

$$\therefore \bar{y} = \frac{\int y\, dV}{\int dV} = \frac{\pi r^4}{4 \times \frac{2}{3} \pi r^3} = \frac{3}{8} r \text{ from base}$$

Ex: Locate the centroid C of the shaded area obtained by cutting a semi circle of dia ‘a’ from the quadrant of a circle of radius ‘a’ as shown.



Sol: Let A_1 – area of quadrant of circle & A_2 – area of semi circle

$$\bar{x} = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2} \quad and \quad \bar{y} = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2}$$

$$A_1 = \frac{\pi a^2}{4} \quad ; \quad A_2 = \frac{\pi a^2}{8}$$

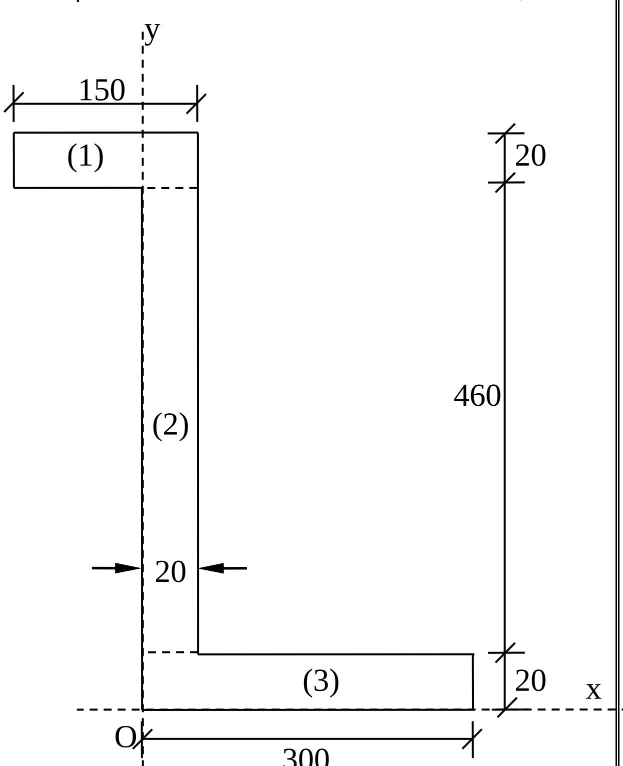
$$A_1 - A_2 = \frac{\pi a^2}{8}$$

$$x_1 = \frac{4a}{3\pi} \;;\; x_2 = \frac{a}{2} \quad and \quad y_1 = \frac{4a}{3\pi} \;;\; y_2 = \frac{2a}{3\pi}$$

$$\bar{x} = \frac{\frac{\cancel{\pi a^2}}{4} \cdot \frac{4a}{3\pi} - \frac{\cancel{\pi a^2}}{8} \cdot \frac{a}{2}}{\frac{\cancel{\pi a^2}}{8}} = \frac{8a}{3\pi} - \frac{a}{2} = 0.349\, a$$

$$\bar{y} = \frac{\frac{\cancel{\pi a^2}}{4} \cdot \frac{4a}{3\pi} - \frac{\cancel{\pi a^2}}{8} \cdot \frac{2a}{3\pi}}{\frac{\cancel{\pi a^2}}{8}} = \frac{8a}{3\pi} - \frac{2a}{3\pi} = 0.636\, a$$

Ex: find the centroid of Z – section shown.(All units are in mm)



Sol: $A_1 = 150 \times 20 = 3000$ $\quad$ $A_2 = 460 \times 20 = 9200$

$A_3 = 300 \times 20 = 6000$

$x_1 = 75 - 20 = 55$ $\quad$ $y_1 = 490$

$x_2 = 10$ $\quad$ $y_2 = 250$

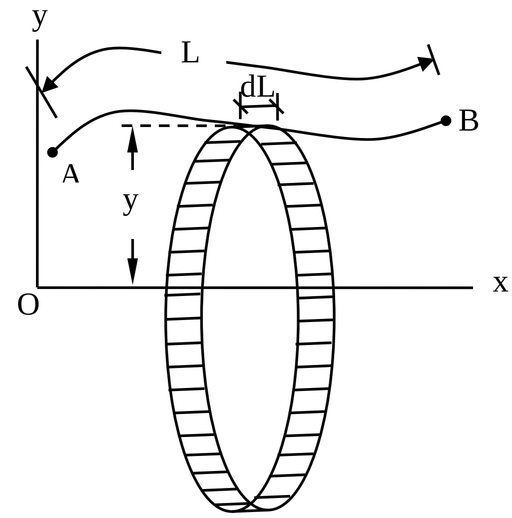
$x_3 = 150$ $\quad$ $y_3 = 10$

$$\therefore \bar{x} = \frac{-(3000 \times 55) + (9200 \times 10) + (6000 \times 150)}{3000 + 9200 + 6000}$$

$$\bar{x} = \frac{827000}{18200} = 45.44\, mm$$

$$\therefore \bar{y} = \frac{(3000 \times 490) + (9200 \times 250) + (6000 \times 10)}{3000 + 9200 + 6000}; \quad \bar{y} = \frac{3830000}{18200} = 210.44\, mm$$

Theorems of pappus and guldinus:



These theorems are useful to determine the surface area or the volume generated by revolving respectively a plane curve or a plane area about a non intersecting axis lying in its plane.

First theorem: It states that the surface area is the product of the length of the generating curve multiplied by the distance travelled by its centroid.

Let AB – curve of length L

Let this curves is resolved about ox through an angle 2π rad.

The differential length dL sweeps through the distance 2πy thereby generating a hoop whose surface area is 2πy dL.

Thus total area of all such loops

$$A = \int 2\,\pi\, y\, dL = 2\,\pi \int y\, dL = 2\,\pi\, \bar{y}\, L$$

$$\text{But } \bar{y} = \frac{\int y\, dL}{\int dL} = \frac{\int y\, dL}{L}$$

$$\therefore \int y\, dL = \bar{y}\,.L$$

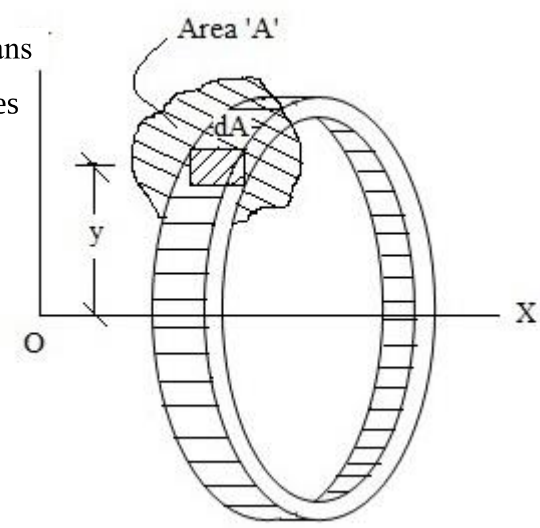
Where L – length of the curve and

2 π ȳ - distance travelled by centroid of the curve

Second theorem: It states that the volume is the product of the figure multiplied by the length of the path described by the centroid of the area.

Let A – total area to be rotated about x-x axis through an angle 2π radians

The differential area dA sweeps through the distance 2 π y and generates a ring whose volume is 2 π y dA.



$$\text{But } \bar{y} = \frac{\int y\, dA}{\int dA} = \frac{\int y\, dA}{A}, \quad \int y\, dA = \bar{y}\,.A$$

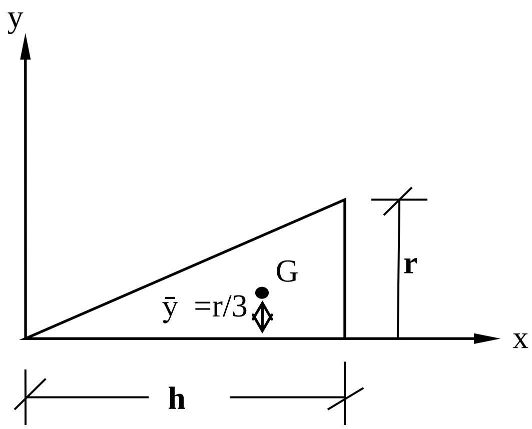
Total volume generated =

$$\int 2\pi\, y\, dA = 2\pi \int y\, dA = 2\pi\, A\, \bar{y} = A\, 2\pi\bar{y}$$

Where A - area of figure

2π ȳ - distance travelled by the centroid of area 'A'

Ex.: Using pappus and Guldinus Theorem , determine the volume of right circular cone the volume generated by rotating the triangle about x-x axis with 2π radians.

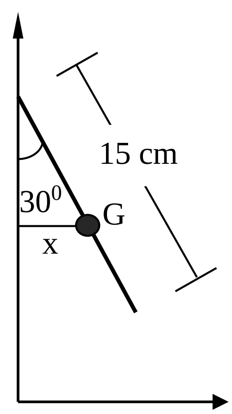


Sol: Volume = Area x distance travelled by C.G for 2 π

$$V = \frac{hxr}{2} x \frac{2\pi r}{3}$$

$$V = \frac{\pi r^2 h}{3}$$

Ex.: Determine the surface area of right circular cone generated by rotating a line about y- axis.



Sol: $\sin 30^0 = \frac{1}{2} = \frac{x}{7.5}$

$$x = \overline{y} = 7.5 \sin 30^0 = 3.75$$

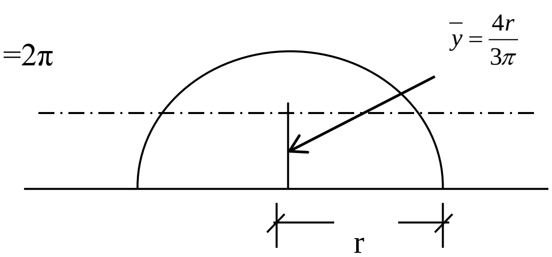
Surface Area = 2π ȳ. L

$$= 2\pi x 15 x 3.75 \ = 353.475 \ cm^2$$

Ex.: Semicircle is rotated about its diameter to generate a sphere .Calculate the volume of sphere

Sol: $A = \frac{\pi r^2}{2}$; $\overline{y} = \frac{4r}{3\pi}$; Angle of revolution =2π

$$V = \frac{\pi r^2}{2} x 2\pi x \frac{4r}{3\pi} = \frac{4\pi r^3}{3}$$



MOMENT OF INERTIA:

Introduction:

In earlier units it is already discussed that the moment of a force (*P*) about a point, is the product of the force and perpendicular distance (*x*) between the point and the line of action of the force (*i.e. P.x*).This moment is also called first moment of force. If this moment is again multiplied by the perpendicular distance (*x*) between the point and the line of action of the force *i.e. P.x* (*x*) = Px^2, then this quantity is called moment of the moment of a force or second moment of force or moment of inertia (briefly written as M.I.). Sometimes, instead of force, area or mass of a figure or body is taken into consideration. Then the second moment is known as second moment of area or second moment of mass. But all such second moments are broadly termed as moment of inertia. In this chapter, the moment of inertia of plane areas are only discussed.

M.I for the plane areas:

<u>Inertia:</u> The property of the matter by virtue of which it resists any change in its state of rest or uniform motion is called as inertia

Translatory motion is identified as mass.

Rotational inertia is termed as moment of inertia.

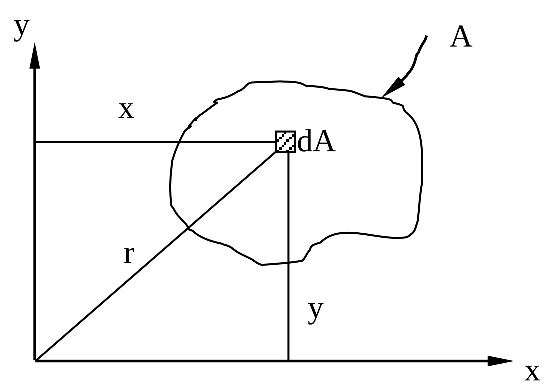
Moment of inertia of any area about any axis is defined as the second moment of area about that axis it's denoted by I.

Let dA be the area of any element situated at a distance of x and y from the axis as shown.

Moment of inertia of A with respect to x and y axis

$$I_{xx} = \int y^2 dA$$

$$I_{yy} = \int x^2 dA$$



Where I_{xx} Moment of inertia of A with respect to x axis

I_{yy} Moment of inertia of A with respect to y axis

But $r^2 = x^2 + y^2$

Integrating and multiplying with dA

$$\int r^2 dA = \int x^2 dA + \int y^2 dA$$

$I_{zz} = I_{xx} + I_{yy}$ Polar moment of inertia

It is also known as perpendicular axis theorem. Thus, the perpendicular axis theorem states that the moment of inertia of an area with respect to an axis perpendicular to x-y plane and passing through origin will be equal to the sum of moments of inertia of same area about x-x and y-y axis. It is termed as the polar moment of inertia.

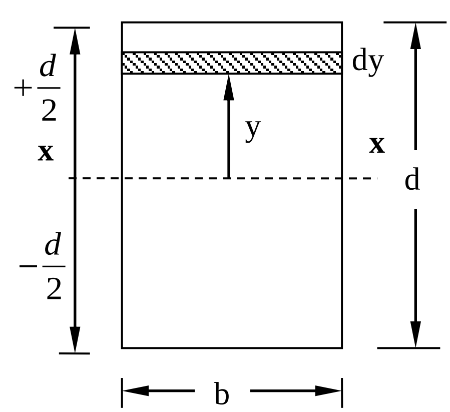
Moment of inertia of a rectangular section:

Consider a rectangular section *ABCD* as shown in Fig.01. whose moment of inertia is required to be found out.

Let b = Width of the section and

d = Depth of the section.

Now consider a elemental strip of thickness dy parallel to X-X axis and located at a distance y from reference axis x-x as shown in the figure



∴ Area of the strip = $b.dy$

We know that moment of inertia of the strip about X-X axis,

= Area × y^2 = $(b.\ dy)\ y^2 = b.\ y^2.\ dy$

Now *moment of inertia of the whole section may be found out by integrating the above equation for the whole length of the lamina *i.e.* from $--\frac{d}{2}$ to $+\frac{d}{2}$,

$$I_{xx} = \int y^2 dA = \int_{-\frac{d}{2}}^{+\frac{d}{2}} y^2 dA = \int_{-\frac{d}{2}}^{+\frac{d}{2}} b.y^2.dy = b\int_{-\frac{d}{2}}^{+\frac{d}{2}} y^2.dy$$

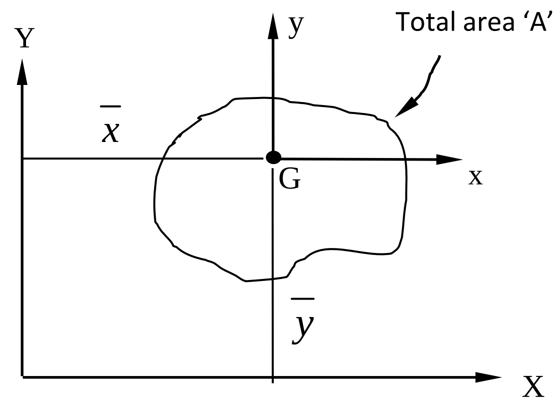
$$= b\left[\frac{y}{3}\right]_{-\frac{d}{2}}^{+\frac{d}{2}} = b\left[\frac{(d/2)^3}{3} - \frac{(-d/2)^3}{3}\right] = \frac{bd^3}{12}$$

$$\therefore I_{xx} = \frac{bd^3}{12}$$

Similarly $\therefore I_{yy} = \frac{db^3}{12}$ $\qquad \therefore I_{zz} = I_{xx} + I_{yy} = \frac{bd^3}{12} + \frac{db^3}{12}$

Parallel axis theorem:

Moment of inertia of plane area with respect any axis in its plane ‘S’ equals to moment of inertia with respect to a parallel centroidal axis plus the product of total area and the square of distance between these two parallel axes



$$I_{XX} = I_{xx} + A\bar{y}^2$$

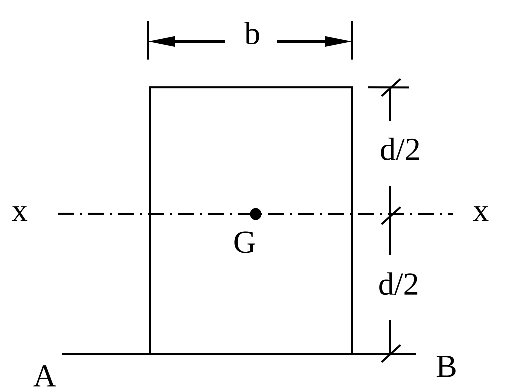
$$I_{YY} = I_{yy} + A\bar{x}^2$$

<u>Moment of inertia of rectangular lamina about axis A-B</u>

$I_{xx} = \frac{bd^3}{12}$; $\bar{y} = \frac{d}{2}$; A=bd

$I_{AB} = I_{xx} + A\bar{y}^2$ $\qquad I_{AB} = \frac{bd^3}{12} + A\left(\frac{d}{2}\right)^2$

$$= \frac{bd^3}{12} + bd\left(\frac{d}{2}\right)^2 = \frac{bd^3}{12} + \frac{bd^3}{4} = \frac{bd^3}{12}(1+3) = \frac{bd^3}{3}$$



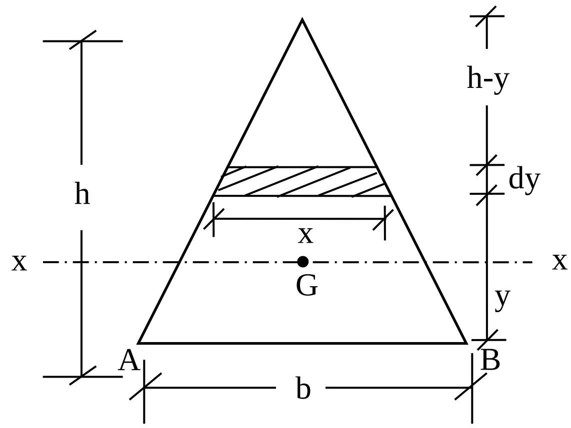
<u>Moment of inertia of a triangular section:</u>

$$I_{AB} = \int y^2 dA$$

$dA = xdy$

By the property of similar triangular, we have

$$\frac{x}{b}=\frac{h-y}{h} \quad = \quad x=b\left(\frac{h-y}{h}\right)$$



Area of elemental strip $dA = \ xdy \ = b\left(\frac{h-y}{h}\right)dy$

From the basic principle of M.I

$$I_{AB}=\int y^2 b\left(\frac{h-y}{h}\right)dy \quad =\frac{b}{h}\int\left(hy^2-y^3\right)dy$$

Integrate with respect to 0 to h

$$I_{AB}=\frac{b}{h}\int_0^h\left(hy^2-y^3\right)dy \quad =\frac{b}{h}\left[\frac{hy^3}{3}-\frac{y^4}{4}\right]_0^h=\frac{b}{h}\left[\frac{h^4}{3}-\frac{h^4}{4}\right]$$

$$\therefore I_{AB}=\frac{bh^3}{12}$$

Applying the parallel axis theorem

$$I_{AB}=I_{xx}+A\bar{y}^2 \qquad \bar{y}=\frac{h}{3} \quad ; \quad A=\frac{bh}{2}$$

$$\frac{bh^3}{12}=I_{xx}+\frac{bh}{2}\left(\frac{h}{3}\right)^2$$

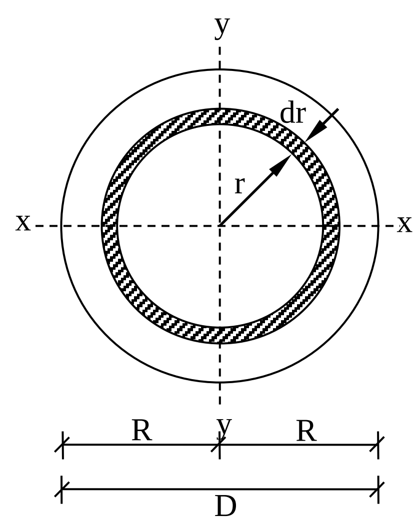
$$I_{xx}=\frac{bh^3}{12}-\frac{bh}{2}\left(\frac{h}{3}\right)^2 \quad ; \quad I_{xx}=\frac{bh^3}{12}-\frac{bh^3}{18} \quad \therefore I_{xx}=\frac{bh^3}{36}$$

Moment of Inertia of Circle:

Consider elemental thin circular ring of width dr and radius r as shown in figure.

Area of ring $dA=2\pi r dr$

From the basic principles polar M.I $I_{zz}=\int r^2 dA$



$$I_{zz}=\int r^2 2\pi r dr$$

$$I_{zz}=2\pi\int r^3 dr$$

Integrate from 0 to R

$$I_{zz}=2\pi\int_0^R r^3 dr \quad , \quad I_{zz}=2\pi\left[\frac{r^4}{4}\right]_0^R \quad =2\pi\frac{R^4}{4}$$

$$\therefore I_{zz}=\frac{\pi R^4}{2}$$

Since circle is symmetric about x-x and y-y axis, we have

$$I_{xx} = I_{yy}$$

By perpendicular axis theorem, we have

$$I_{zz} = I_{xx} + I_{yy} = 2I_{xx} \quad ; \quad \therefore I_{xx} = \frac{I_{zz}}{2} \quad \text{and} \quad I_{yy} = \frac{I_{zz}}{2}$$

$$\therefore I_{xx} = I_{yy} = \frac{I_{zz}}{2} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}$$

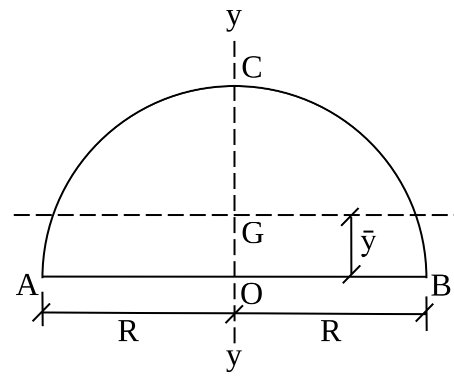
M.I of semicircle about diameter:

Axis of base coincide with centroidal axis of a circle

$$\therefore I_{AB} = \frac{I_{xx}}{2} = \frac{\pi R^4}{8}$$

Similarly

$$\boxed{\therefore I_{OC} = \frac{I_{yy}}{2} = \frac{\pi R^4}{8}}$$



By parallel axis theorem $\quad I_{AB} = I_{xx} + A\bar{y}^2$

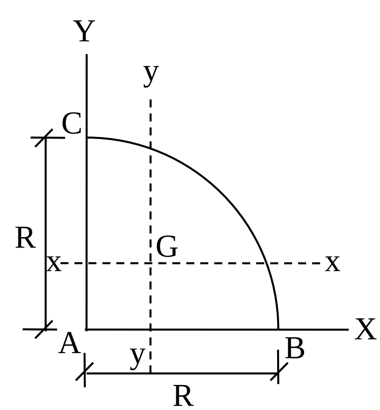
Here $\quad I_{AB} = \dfrac{\pi R^4}{8} \quad ; \quad \bar{y} = \dfrac{4r}{3\pi} \quad ; \quad A = \dfrac{\pi R^2}{2}$

$$I_{xx} = I_{AB} - A\bar{y}^2 = \frac{\pi R^4}{8} - \left(\frac{4r}{3\pi}\right)^2 \frac{\pi R^2}{2}$$

$$\boxed{I_{xx} = 0.11R^4}$$

[M.I about parallel centroidal axis to diameteral axis]

M.I of quarter circle about its base:



I_{AB} is about an axis of a circle

$$I_{AB} = \frac{\pi R^4}{4} * \frac{1}{4} = \frac{\pi R^4}{16} \quad \text{Similarly} \quad I_{AC} = \frac{\pi R^4}{16}$$

By parallel axis theorem $\quad I_{AB} = I_{xx} + A\bar{y}^2$

Here $\quad I_{AB} = \dfrac{\pi R^4}{16} \quad ; A = \dfrac{\pi R^2}{4} \quad ; \ \bar{y} = \dfrac{4r}{3\pi}$

$$I_{xx} = I_{AB} - A\bar{y}^2 = \frac{\pi R^4}{16} - \frac{\pi R^2}{4}\left(\frac{4R}{3\pi}\right)^2$$

$$= R^4\left[\frac{\pi}{16} - \frac{\pi}{4}\left(\frac{4}{3\pi}\right)^2\right] = 0.055R^4$$

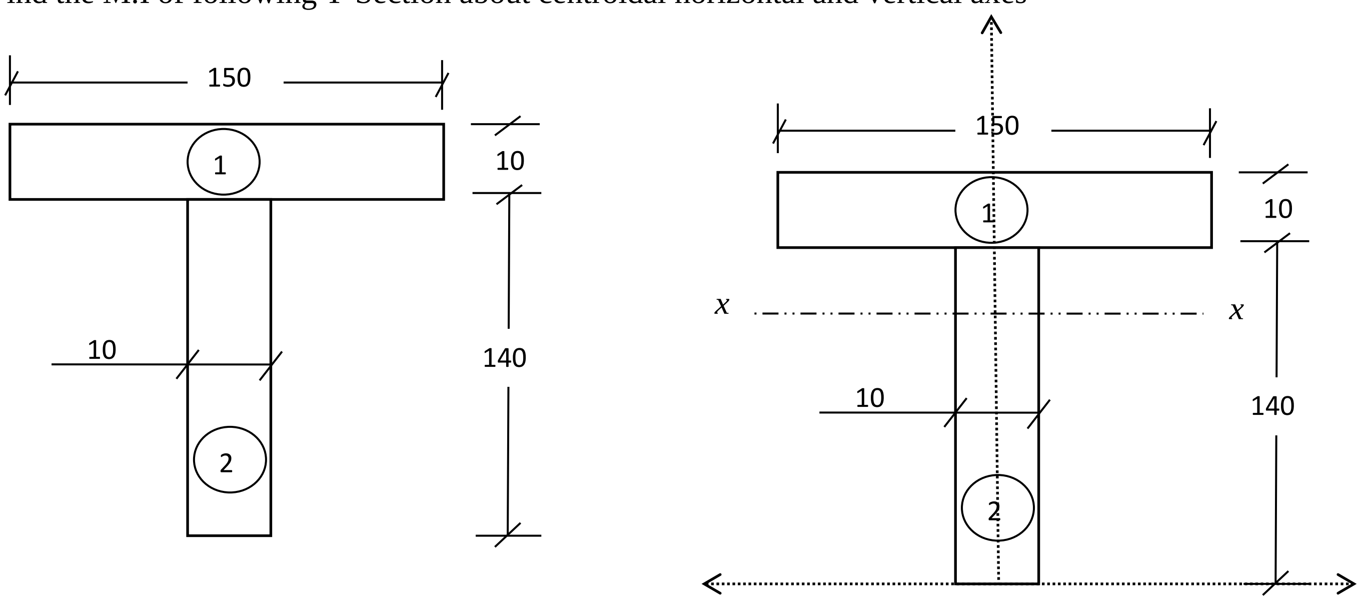
M.I of composite Sections:

Beams and columns having composie sections are commonly used as structural elements. A composites area consists of connected similar parts or shapes, such as rectangular, triangle, circle, semicircle and quarter circles. By knowing the M.I of the each section, the M.I of composite area is calculated. The moment of inertia of the composite area about an axis equals the algebraic sum of the moments of inertia of all its parts about same axis.

Procedure of M.I of composite section:

- Divide the given composite section into number of simple standard areas such as triangle, circle, semicircle and quarter circles
- Determine the moment of inertia of each part about its centroidal axis
- Calculate moment of inertia of each part about given axis, then algebraic sum of moment of inertia of each part about given axis gives M.I of composite section about that axis.
- If composite section consists may removed areas, then the moment of inertia of removed area should be subtracted from the M.I of composite section.

Ex: Find the M.I of following T-Section about centroidal horizontal and vertical axes



Sol: Centroid of composite section

$\bar{x} = 0$ because section is symmetric about y-axisq

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$a_1 = 150x10$; $y_1 = 145$

$a_2 = 140x10$; $y_2 = 70$

$$\bar{y} = \frac{(150 \times 10 \times 145) + (140 \times 10 \times 70)}{(150 \times 10 + 140 \times 10)}$$

$$\bar{y} = 108.79mm$$

For Area: 1

Moment of inertia of area -01 about its own centroidal axis is = $I_1 = \frac{bd^3}{12}$

Moment of inertia of area-01 about centraoidal axis of composite section,

By uing parallel axis theorem By parallel axis theorem : $I_{x1} = I_1 + A_1\left(\overline{y} - y_1\right)^2$

$$I_{x1} = \frac{150 \times 10^3}{12} + (150 \times 10)(108.79 - 145)^2$$

For Area-02:

Moment of inertia of area -01 about its own centroidal axis is = $I_2 = \frac{bd^3}{12}$

Moment of inertia of area-01 about centraoidal axis of composite section,

By uing parallel axis theorem by parallel axis theorem : $I_{x2} = I_2 + A_2\left(\overline{y} - y_2\right)^2$

$$I_{x2} = \frac{10x140^3}{12} + (140x10)(108.79 - 70)^2$$

Moment of inertia about x-x axis: $I_{xx} = I_{x1} + I_{x2}$

$$I_{xx} = \left(\frac{150 \times 10^3}{12} + (150 \times 10)(108.79 - 145)^2\right) + \left(\frac{10 \times 140^3}{12} + (140 \times 10)(108.79 - 70)^2\right)$$

$$I_{xx} = (12500 + 1921386.15) + (2286666.66 + 2106529.74)$$

$$I_{xx} = (12500 + 1921386.15) + (2286666.66 + 2106529.74)$$

$$I_{xx} = 6327082.55mm^4$$

Similarly M.I of composite section about centroidal y-y axis: $I_{yy} = I_{y1} + I_{y2}$

$$I_{y1} = \frac{10x150^3}{12} + 0 = \frac{10x150^3}{12} \; ; \qquad I_{y2} = \frac{140x10^3}{12} + 0 = \frac{140x10^3}{12}$$

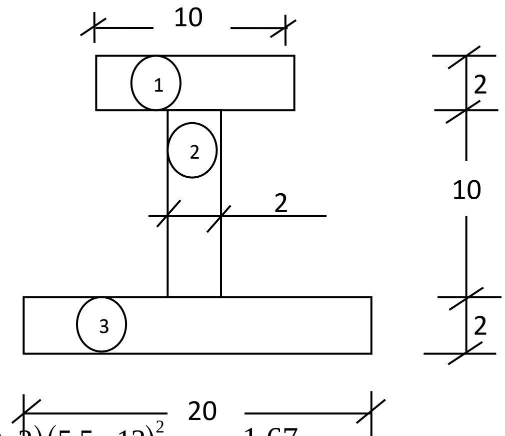
$$I_{yy} = \frac{10x150^3}{12} + \frac{140x10^3}{12}$$

$$I_{yy} = 2812500 + 11666.66$$

$$I_{yy} = 2824166.667mm^4$$

Ex: Find the M.I of following I-Section about centroidal horizontal and vertical axes. All dimension are in centimetre.

Sol:



Centroid of I-Section: $\bar{y} = \dfrac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$

$$\bar{y} = \frac{(10x2)13 + (10x2)7 + (20x2)1}{(10x2) + (10x2) + (20x2)} = \frac{440}{80} = 5.5cm$$

<u>Area-01</u>

M.I of area-01 about its own centroidal axis: $I_1 = \dfrac{bd^3}{12} = \dfrac{10x2^3}{12}$

M.I of area-01 about centroidal axis x-x of I –Section: $I_{x1} = \dfrac{10x2^3}{12} + (10x2)(5.5-13)^2 = 1131.67$

M.I of area-02 about centroidal axis x-x of I –Section: $I_{x2} = \dfrac{2x10^3}{12} + (10x2)(5.5-7)^2$ =211.67

M.I of area-01 about centroidal axis x-x of I –Section: $I_{x3} = \dfrac{20x2^3}{12} + (20x2)(5.5-1)^2$ =823.33

M.I of I-Section about centroidal horizontal axis x-x is: $I_{xx} = I_{x1} + I_{x2} + I_{x3}$

$$I_{xx} = I_{x1} + I_{x2} + I_{x3} = 1131.67 + 211.67 + 823.33 = 2166.67\ cm4$$

M.I of I-Section about centroidal horizontal axis x-x is : $I_{yy} = I_{y1} + I_{y2} + I_{y3}$

$$I_{yy} = \frac{2x10^3}{12} + 0 + \frac{10x2^3}{12} + 0 + \frac{2x20^3}{12} + 0 = 116.67 + 6.67 + 1333.33 = 1506.67cm^4$$

Ex: Find the M.I of following Section about x-x axis. All dimensions are in centimetre

Sol:

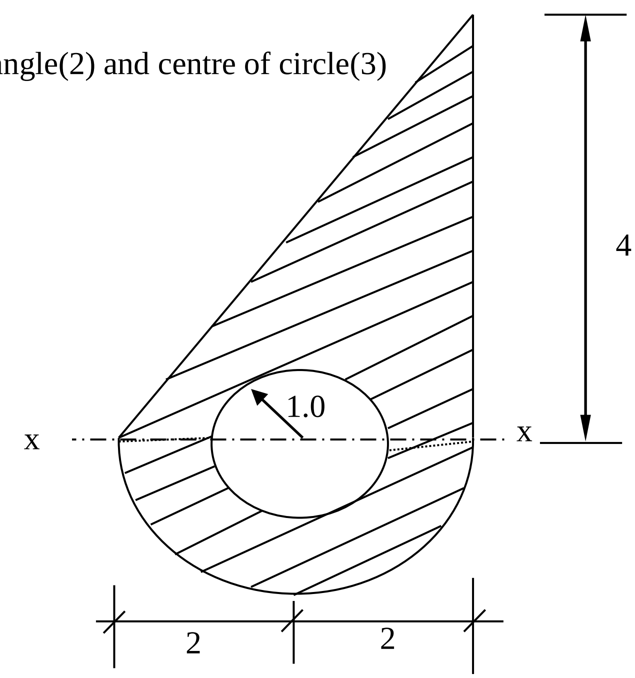
x-x axis is common axis for base of semicircle (1), base of triangle(2) and centre of circle(3)

$$I_{xx} = I_{x1} + I_{x2} - I_{x3}$$

$I_{x1} = \dfrac{\pi x2^4}{8}$; $I_{x2} = \dfrac{4x4^3}{12}$; $I_{x3} = \dfrac{\pi x1^4}{4}$

$$I_{xx} = \frac{\pi x2^4}{8} + \frac{4x4^3}{12} - \frac{\pi x1^2}{4} = 6.28 + 21.33 - 0.79$$

$$I_{xx} = 26.82cm^4$$



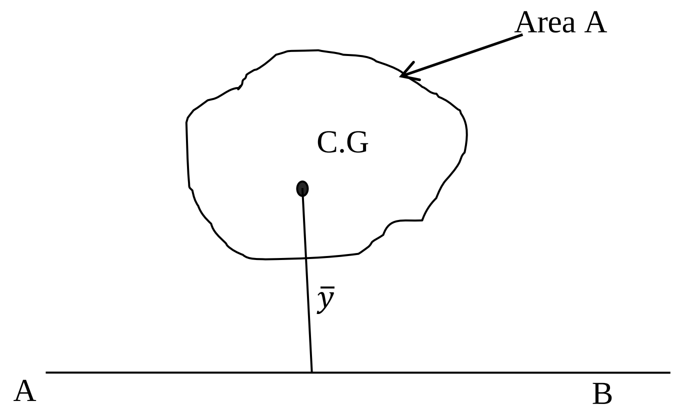
Radius of gyration:

It is mathematical term defined by the relation $k = \sqrt{\dfrac{I}{A}}$

A - Area of the figure (cross sectional area)

I - Moment of inertia

K - Radius of gyration

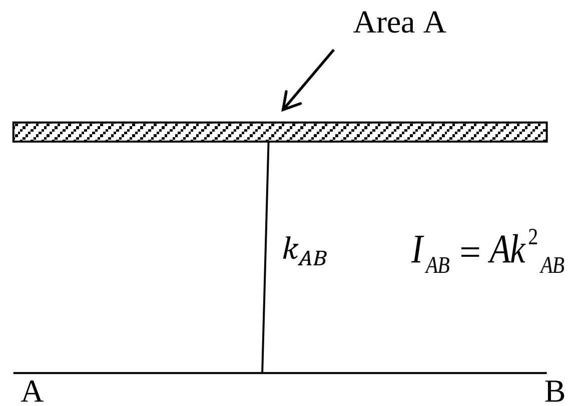


$k_{xx} = \sqrt{\dfrac{I_{xx}}{A}}$ Radius of gyration about x-axis

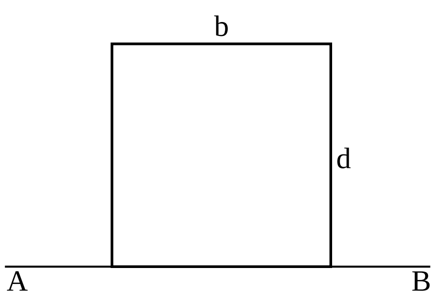
$k_{yy} = \sqrt{\dfrac{I_{yy}}{A}}$ Radius of gyration about y-axis

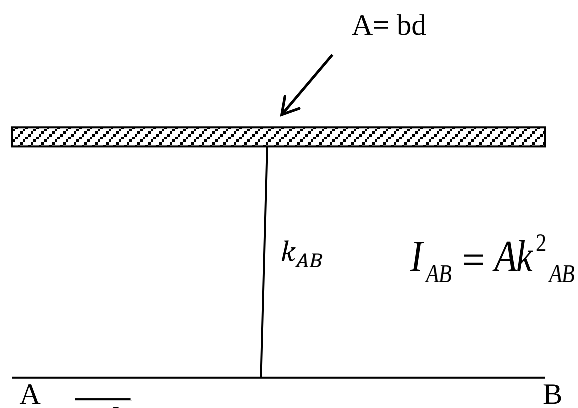
$k_{AB} = \sqrt{\dfrac{I_{AB}}{A}}$ Radius of gyration about AB-axis

Radius of gyration can also be considered as the distance at which the total area is distributed in form of strip of negligible width as shown in figure, such that there is no change in moment of inertia.



Ex:



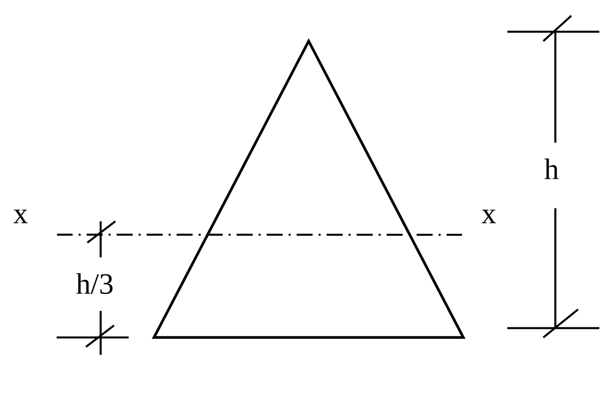


$I_{AB} = \dfrac{bd^3}{12}$; Radius of gyration about AB-axis $k_{AB} = \sqrt{\dfrac{\frac{bd^3}{12}}{bd}} = \dfrac{d}{2\sqrt{3}}$

Ex: Radius of gyration of a triangle about its centroidal axis

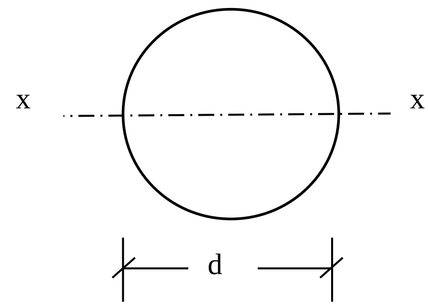
$I_{xx} = \dfrac{bh^3}{36}$ A = $\dfrac{bh}{2}$; $k_{xx} = \sqrt{\dfrac{I_{xx}}{A}}$ = $\sqrt{\dfrac{\frac{bh^3}{36}}{\frac{bh}{2}}}$

= $\sqrt{\dfrac{h^2}{18}} = \dfrac{h}{3\sqrt{2}}$



$\therefore k_{xx} = \dfrac{h}{3\sqrt{2}}$

Ex:03 Determine the radius of gyration of circular disc whose diameter is 'd'.



$$I_{xx} = \frac{\pi d^4}{64} \quad ; \; A = \frac{\pi d^2}{4}$$

$$k_{xx} = \sqrt{\frac{I_{xx}}{A}} = \sqrt{\frac{\frac{\pi d^4}{64}}{\frac{\pi d^2}{4}}} = \frac{d}{4} \qquad \therefore k_{xx} = \frac{d}{4}$$

UNIT –IV

Objective:

Mass moment of inertia is the rotational analogue of mass in linear motion and must be specified with respect to a chosen axis of rotation.

Method of virtual work is an alternative approach to analyze the equilibrium of any loaded system and the unknown forces can be determined. For an equilibrium force system, the algebraic summation of work done by all effective force components is equals to zero. Virtual work principle can mainly applicable to simple beams, trusses, connected systems and ladder problems.

Syllabus:

Mass Moment of Inertia: Centre of mass definition, moment of inertia-definition, transfer formulae for mass moment of inertia, mass moment of inertia for standard bodies.

Learning Outcomes:

Student will be able to

- Derive the expressions for mass moment of inertia of standard bodies like rectangular and triangular laminas, cone, cylinder, sphere etc.. from first principles
- Understand the application of parallel axis theorem for obtaining mass moment of inertia values about any axis parallel to its centroidal axis
- Calculate the mass moment of inertia of composite bodies.
- Apply for obtaining the efficiency of screw jack and other systems acting on horizontal or inclined planes and lifting machines

Learning material

Mass moment of inertia: The second moment of mass about any axis is known as mass moment of inertia.

$$I_{xx} = \int y^2 \ dm \quad ; \quad I_{yy} = \int x^2 \ dm$$

$$I_{zz} = \int r^2 \ dm = 2\int x^2 \ dm = 2\,I_{xx} = 2\,I_{yy}$$

Units of mass moment of inertia (I):

Unit of mass: $\dfrac{N - sec^2}{m}$ (kg)

mass $\times$ distance^2 $= \dfrac{N - sec^2}{m} \times m^2 = N - sec^2 - m$

Unit of mass moment of inertia is N – sec^2 – m

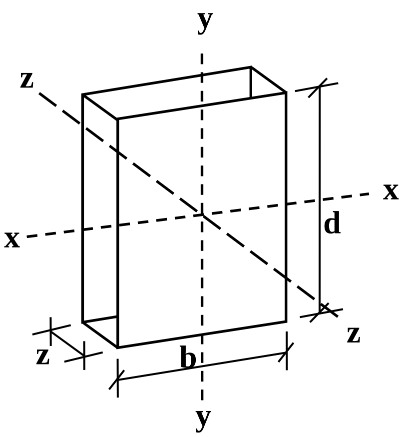
Mass = density × volume = ρ. Area × thickness

For uniform thick laminas,

Mass = ρ t A = ρ t (Area) where 'ρ t' is constant.

Mass moment of inertia of a rectangular lamina:

Mass, M = ρ t b d



We know that $I_{xx} = \int y^2 \ dA$

For a rectangular lamina

$$I_{xx} = \rho\, t \int y^2 \ dA$$

Similarly $I_{yy} = \rho\, t \int x^2 \ dA$ and $I_{zz} = \rho\, t \int r^2 \ dA$

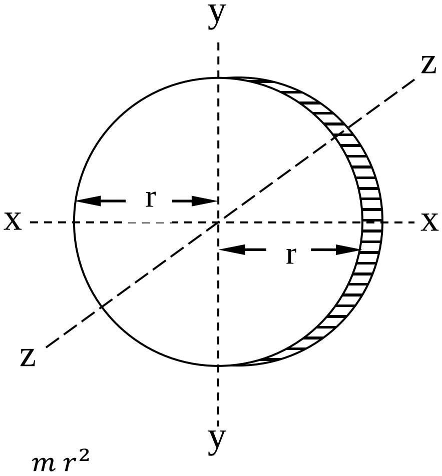
$$I_{xx} = \rho\, t \int y^2 \ dA = \rho\, t \left[\frac{bd^3}{12}\right] = \rho\, t\, b\, d \left(\frac{d^2}{12}\right)$$

$$\therefore I_{xx} = m\left(\frac{d^2}{12}\right) \ similarly\ I_{yy} = m\left(\frac{b^2}{12}\right)$$

$$I_{zz} = \frac{m}{12}\,(b^2 + d^2)$$

Mass moment of inertia of a circular disc:

$I_{xx} = I_{yy}$



For circle, $I_{xx} = \dfrac{\pi\, r^4}{4}$

Mass of disc = ρ t. π r^2

Mass moment of inertia

$$I_{xx} = \rho\, t \int y^2 \ dA = \rho\, t \left[\frac{\pi r^4}{4}\right] = \rho\, t\, \pi r^2 \left(\frac{r^2}{4}\right)$$

$$\therefore I_{xx} = \frac{m\, r^2}{4} \ ; \ I_{yy} = \frac{m\, r^2}{4}$$

Mass moment of inertia about polar axis $J = I_{zz} = I_{xx} + I_{yy} = \dfrac{m\, r^2}{2}$

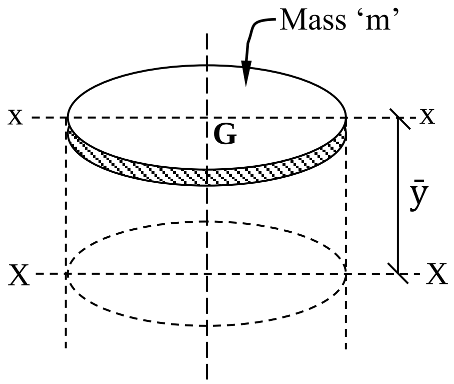
Use of parallel axis theorem

$$I_{XX} = I_{xx} + m\,\bar{y}^2$$

Where I_{XX} = Mass moment of inertia about parallel axis XX

I_{xx} - Mass moment of inertia of disc about its C.G

$\bar{y}$ - distance between two parallel axis



Mass moment of inertia of a right circular cylinder:

Plane axis are xx and yy, polar axis is z – z.

To calculate the mass M.I about x – axis, take an element disc of radius ‘r’ and thickness ‘dz’ at a distance ‘z’ from x – x axis.

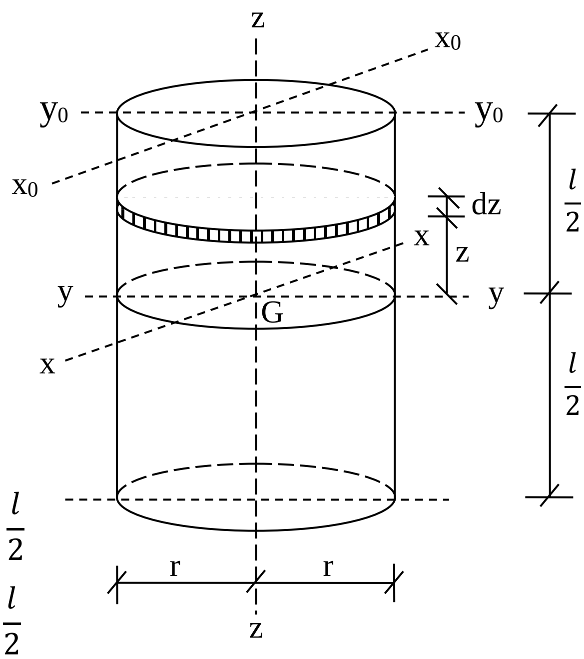
Mass of disc = $\pi r^2 \rho.\, dz$

Mass moment of inertia of the disc about x – x axis is

Applying parallel axis theorem,

For element, $I_{xx} = (\pi r^2 . \rho\, dz)\dfrac{r^2}{4} + \pi r^2 . \rho\, dz\,(z^2)$

For the entire cylinder,



$$I_{xx} = \pi r^2 \rho \int_{-\frac{l}{2}}^{+\frac{l}{2}} \left(\frac{r^2}{4} + z^2\right) dz = \pi r^2 \rho \left[\left(\frac{r^2}{4} z\right) + \left(\frac{z^3}{3}\right)\right]_{-\frac{l}{2}}^{+\frac{l}{2}}$$

$$= \pi r^2 \rho \left[\left(\frac{r^2\, l}{4}\right) + \left(\frac{l^3}{24} - \left(-\frac{l^3}{24}\right)\right)\right] = \pi r^2 \rho \left[\frac{r^2\, l}{4} + \frac{l^3}{12}\right] = \pi r^2 \rho . l \left[\frac{r^2}{4} + \frac{l^2}{12}\right]$$

But $\pi r^2 l.\, \rho$ is the mass of the cylinder = m

$$I_{xx} = m\left[\frac{r^2}{4} + \frac{l^2}{12}\right] \qquad and \quad I_{yy} = m\left[\frac{r^2}{4} + \frac{l^2}{12}\right]$$

$$I_{zz} = m\left[\frac{r^2}{2} + \frac{l^2}{6}\right]$$

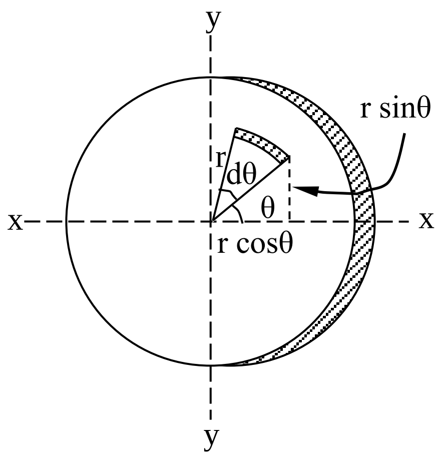
Mass moment of inertia of a disc from first principles whose radius is ‘R’ and thickness ‘t’ about its centroidal axis:

Consider element area (r. dθ) dr and thickness ‘t’

Let ρ be the mass density

dm = ρ. r dθ. dr. t = (ρ t) r dθ. dr

Distance of dm from x – x axis = r sinθ



$$\therefore\; I_{xx} = \int (r\, sin\theta)^2 . dm = \int_0^R \int_0^{2\pi} r^2\, sin^2\theta\, .(\rho\, t) r . d\theta\, . dr$$

$$= \rho\, t \int_0^R \int_0^{2\pi} r^3 \left(\frac{1-\cos 2\theta}{2}\right) . d\theta\, . dr$$

$$= \rho\, t \int_0^R \frac{r^3}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} .dr \;=\; \rho\, t \int_0^R \frac{r^3}{2}\,[\,2\pi\,].dr = \pi\,\rho\, t \int_0^R r^3\, dr$$

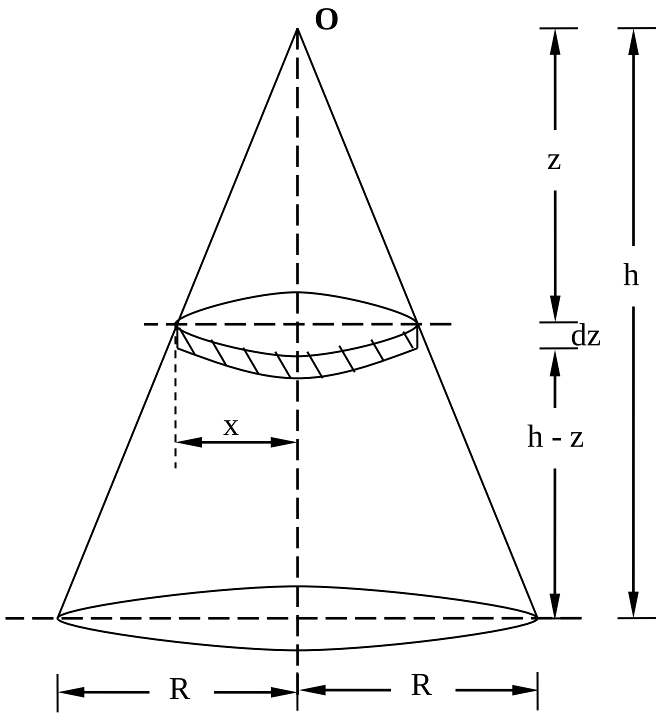
$$= \pi\,\rho\, t\, \left[\frac{r^4}{4}\right]_0^R = \rho\, t\,.\frac{\pi\, R^4}{4}$$

But total mass M = $\pi\, R^2.\ \rho\ t$

$$\therefore\ I_{xx} = I_{yy} = \frac{M\,R^2}{4}\quad ;\quad I_{zz} = \frac{M\,R^2}{2}$$

Mass moment of inertia of a right circular cone about its axis of rotation:

Take an element plate of radius ‘x’ and thickness ‘dz’ at a distance ‘z’ from ‘O’.



dm = $\rho\, \pi\, x^2.\ dz$

Moment of inertia of plate about z –z axis

$$I_{zz} = \frac{m\,x^2}{2} = \frac{\rho\,\pi\,x^2}{2}.dz.x^2 = \frac{\rho}{2}.\pi\,x^4.dz$$

From similar triangles

$$\frac{x}{R} = \frac{z}{h}\quad ;\quad x = \frac{R}{h}.z$$

$$I_{zz}\text{of element plate} = \frac{\rho}{2}.\pi\,\left(\frac{R}{h}\right)^4.z^4.dz$$

Moment of inertia of cone about z – z axis,

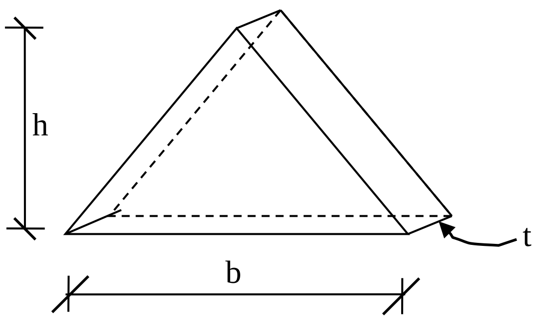
$$I_{zz} = \int_0^h \frac{\rho}{2}.\pi\,\left(\frac{R}{h}\right)^4.z^4.dz = \frac{\rho}{2}.\pi\,\left(\frac{R}{h}\right)^4\left[\frac{z^5}{5}\right]_0^h$$

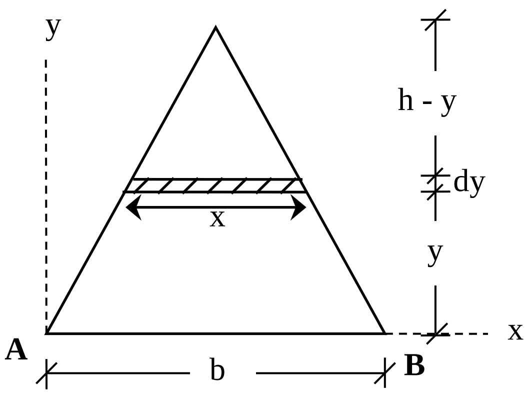
$$= \frac{\rho}{2}.\pi\,\left(\frac{R}{h}\right)^4.\frac{h^5}{5} = \frac{\rho\,\pi\,R^4 h}{10}$$

$$\text{Mass of cone} = \int_0^h dm = \int_0^h \rho\,\pi\,x^2\,dz = \rho\,\pi\int_0^h \frac{R^2}{h^2}\,z^2\,dz = \rho\,\pi\,\frac{R^2}{h^2}\left[\frac{z^3}{3}\right]_0^h = \frac{1}{3}\,\rho\,\pi\,R^2\,h$$

$$\therefore I_{zz} = \left(\frac{1}{3}\,\rho\,\pi\,R^2\,h\right)R^2.\frac{3}{10} = \frac{3}{10}\,MR^2$$

Mass moment of inertia of a homogeneous triangle plate of weight ‘W’ with respect to its base:





Mass of triangle = $\rho\ \frac{1}{2}\ bh.t$

Element mass dm = x. dy. ρ. t (from similar triangles x= $\frac{b}{h}\ (h-y)$)

Mass moment if inertia of dm about base = y^2. (x. dy. ρ.t)

Mass moment of inertia of triangular plate about its base

$$I_{AB} = \int_0^h \frac{b}{h}.(h-y).y^2.\rho t.dy \ = \frac{b}{h}\ \rho t \int_0^h [hy^2 - y^3]\ dy$$

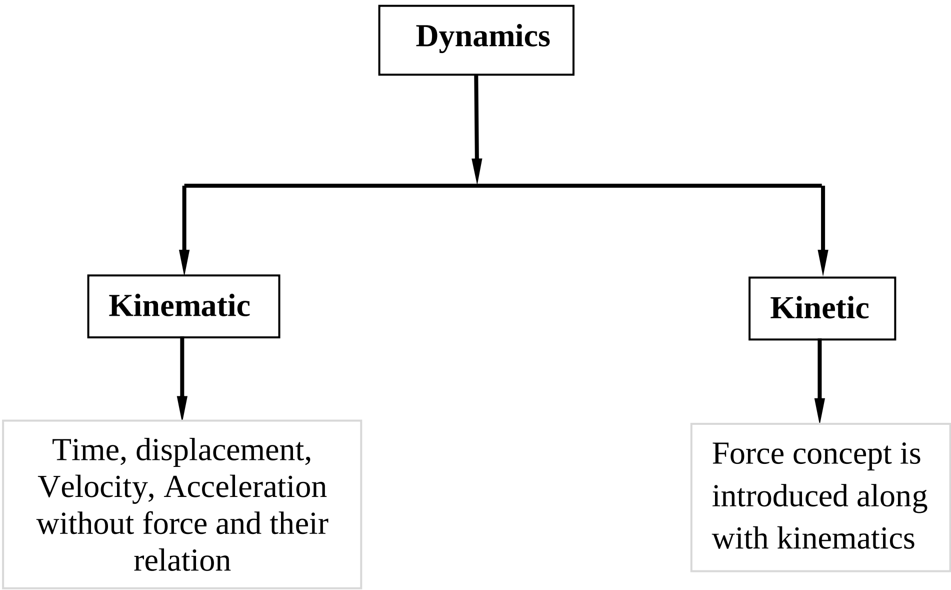
$$= \frac{b}{h}\ \rho t\ \left[\frac{hy^3}{3} - \frac{y^4}{4}\right]_0^h = \frac{b}{h}\ \rho t\ \left[\frac{h^4}{3} - \frac{h^4}{4}\right]$$

$$= \frac{b}{h}\ \rho t\ .h^4.\left[\frac{1}{12}\right] = \frac{\rho t.bh^3}{12}$$

But total mass of triangular plate = m = $\frac{1}{2}\ \rho t.bh$

$$I_{AB} = \left(\frac{1}{2}\ \rho t.bh\right)\frac{h^2}{6} = \frac{mh^2}{6} \qquad but\ m = \frac{W}{g}$$

$$\therefore I_{AB} = \frac{Wh^2}{6g}$$



Types of motions:

1. **Linear or Rectilinear or Translation motion** :

   If the motion of the particle is in a straight line, it is known as linear motion

2. **Curvilinear motion**: if the path of a motion of particle or body is curve ,it is curvilinear motion

   Ex: a car taking a turn on road, projectile

3. **Rotary Motion** :when a body is rotating about a fixed axis, it is rotary motion

   Ex: ceiling fan, electric motor etc,

Linear Motion:

Displacement: the change of position of a moving body with respect time in particular linear direction is its displacement

Ex:$s, ds, x, dx, y, dy..etc.$

Velocity: Rate of change of displacement of a body in straight line is the velocity

$$velocity = \frac{displacement}{time} = \frac{s}{t} = \frac{ds}{dt} = \frac{dx}{dt} = \frac{dy}{dt}$$

Acceleration: Rate of change of increase in velocity of the body is acceleration.

$$acceleration = \frac{velocity}{time} = \frac{v}{t}$$

Deceleration: Rate of change of decrease in velocity of the body is deceleration.

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}; \frac{d^2x}{dt^2}; \frac{d^2y}{dt^2}$$

$$Here \quad \frac{dv}{dt} = \dot{v} \qquad \frac{d^2s}{dt^2} = \ddot{s}; \qquad \frac{d^2x}{dt^2} = \ddot{x}; \qquad \frac{d^2y}{dt^2} = \ddot{y}$$

Equation of motion in straight line with uniform acceleration:

Let v_0 be the initial velocity, 'v' be the final velocity and't' be the time interval, then

Acceleration: $a = \frac{v - v_0}{t}$

$v = v_0 + at \quad --- \ (1)$

$s = average\ velocity\ \times time = \left(\frac{v_0 + v}{2}\right) t = \left(\frac{v_0 + at + v_0}{2}\right) t$

$s = v_0 t + \frac{1}{2} a t^2 \quad --- \ (2)$

$Also\ from\ equation\ (1) \dots\dots\ t = \frac{v - v_o}{a}$

$Also\ from\ equation\ (2) \dots\dots\ s = v_0 \left(\frac{v - v_o}{a}\right) + \frac{1}{2} a \left(\frac{v - v_o}{a}\right)^2$

$s = \left(\frac{v - v_o}{2a}\right)(2v_o + v - v_o) = \left(\frac{v - v_o}{2a}\right)(v_o + v) = \frac{v^2 - {v_o}^2}{2a}$

$v^2 - {v_o}^2 = 2as \quad --- \ (3)$

Equation of motion in straight line with variable acceleration:

$a = \frac{dv}{dt} = v\frac{ds}{dt} \qquad here \quad \therefore v = \frac{ds}{dt}$

Ex: Burglar’s car had a start with an acceleration of 2 m/sec^2. A police vigilant party came after 10 sec and continued to chase the burglar’s car with uniform velocity of 40 m/sec .Find the time taken for the van to overtake the car.

Sol: Let t – Time taken for the van to overtake the car

For burglar’s car:

v_o = 0, a = 2 m/s^2

Time travelled by bungler’s car = t + 10

$\therefore s = v_0 t + \frac{1}{2} a t^2$

$s = 0 + \frac{1}{2} \times 2 \times (t + 10)^2 = t^2 + 20t + 100 \qquad --- (1)$

For police van:

s = uniform velocity × time = 40 × t ---(2)

Equating (1) and (2)

$\therefore\ t^2 + 20t + 100 = 40\,t$

$t^2 - 20t + 100 = 0$

$(t - 10)^2 = 0$

t = 10 seconds.

Ex: A car is moving with velocity of 15 m/sec .the car is brought to rest by applying brakes in 5 sec .Determine

a) The retardation

b) Distances travelled by car after brakes are applied.

Sol: $v = v_0 + at$

0 = 15 + a(5)

a = - 3 m/s^2 (retardation)

$s = v_0 t + \frac{1}{2} a t^2 = (15 \times 5) + \frac{1}{2}(-3)(5)^2 = 70 - 37.5 = 37.5\ m$

Distance travelled in n^{th} second:

$$s = v_0 + \frac{a}{2}(2n - 1)$$

Ex: A body is moving with uniform acceleration and covers 15m in 5^{th} second 25m in 10^{th} second. Determine

a) Initial velocity of the body

b) Acceleration of the body

Sol: $15 = v_0 + \frac{a}{2}\left((2 \times 5) - 1\right) = v_0 + \frac{a}{2}.9$

$15 = v_0 + 4.5\ a$ ----(1)

$25 = v_0 + \frac{a}{2}\left((2 \times 10) - 1\right) = v_0 + \frac{a}{2}.19$

$25 = v_0 + 9.5\ a$ ----(2)

$15 = v_0 + 4.5\ a$ ----(1)

$10 = 5\ a$

$a = 2\ m/s^2$; $v_o = 15 – 9 = 6$ m/s.

Ex: Find the height of the tower from top of which an object falls freely and during the last second of its motion ,the object travels a distance is equal to 2/3 of height of the tower .g =9.81 m/sec^2

Sol: Distance in last one second, $s = \frac{2}{3}\ h$ $----(1)$

Let t – time taken to travel the distance ‘h’

$h = v_0 t + \frac{1}{2} g t^2\ ; v_o = 0$

$\therefore h = \frac{1}{2}\ 9.8\ t^2 = 4.9\ t^2$ $----(2)$

Distance travelled in (t – 1) sec’s is

$h' = v_0(t-1) + \frac{1}{2} g (t-1)^2 \quad ; \ v_o = 0$

$\therefore h' = \frac{1}{2}\ 9.8\ (t-1)^2 = 4.9\ (t-1)^2$ $----(3)$

$Also\ h - h' = \frac{2}{3}\ h$

$\therefore 4.9\ t^2 - \ 4.9\ (t-1)^2 = \frac{2}{3}\ h$

$4.9\ t^2 - \ 4.9\ t^2 + 9.8\ t + 4.9 = \frac{2}{3} \times 4.9\ t^2$

$2t - 1 = \frac{2}{3}\ \ t^2 \quad or \quad t^2 - \ 3t + 1.5 = 0$

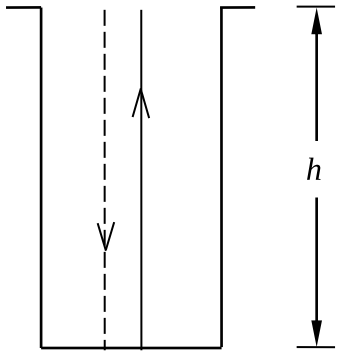
t = 2.366 sec

$\therefore h = 4.9\ t^2 = 4.9 \times\ 2.366^2 = 27.43\ m$

Ex: A stone dropped into a well is heard to strike the water after 4 seconds. Find the depth of the well, if the velocity of sound is 350 m/sec.

Sol: Downward direction:

Initial velocity , $v_o = 0$



Let t_1 – time taken to reach the bottom

t_2 – time taken by the sound to reach the top

v_s = 350 m/sec

$t_1 + t_2 = 4$ sec

$h = v_0 t_1 + \frac{1}{2} g t_1^2 = 0 + \frac{9.81}{2} t_1^2$

$h = 4.905\, t_1^2$ ---(1)

<u>Upward motion:</u>

$h = v_s t_2$

$t_2 = 4 - t_1$

$h = 350\,(4 - t_1) = 1400 - 350\, t_1$ --------- (2)

Equating (1) and (2)

$4.905\, t_1^2 = 1400 - 350\, t_1$

$t_1^2 + 71.35\, t_1 - 285.42 = 0$

By solving the above equation we get $t_1 = 3.8$ sec

From (1) $h = 4.905 \times 3.8^2 = 70.82$ m.

$\therefore h = 70.82\, m.$

<u>Curvilinear Motion</u>: when a moving particle describes a curved path, it is said to have curvilinear motion

Displacement-Time Equations are:

$$s = f(t); \quad x = f_1(t); \quad y = f_2(t)$$

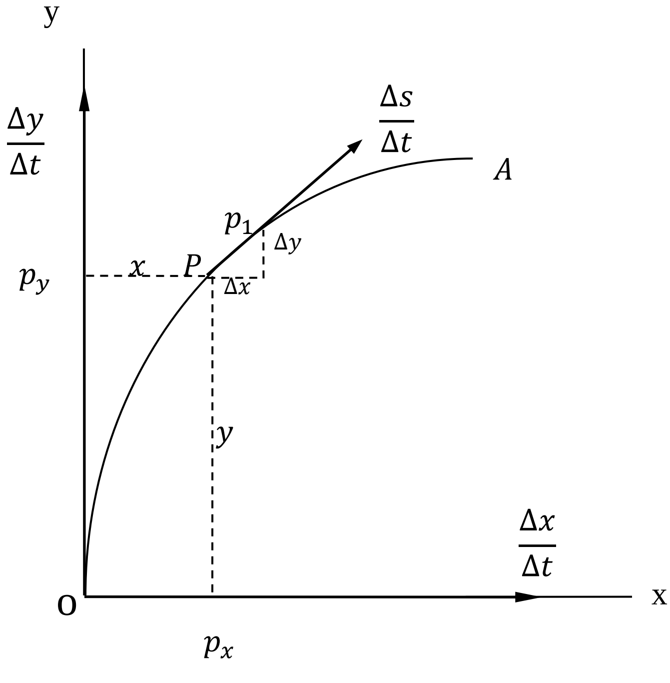
$$(v_{av})_x = \frac{\Delta x}{\Delta t}$$

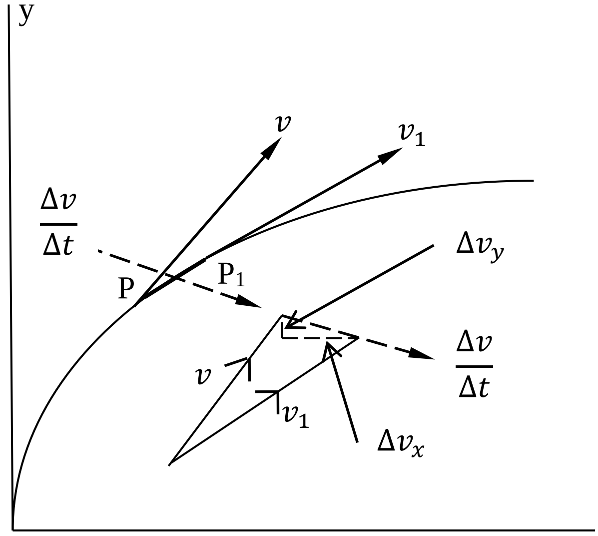
$$v_x = \lim_{\Delta t \to \infty} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x}$$

$$v_y = \lim_{\Delta t \to \infty} \frac{\Delta y}{\Delta t} = \frac{dy}{dt} = \dot{y}$$

$$v = \frac{ds}{dt} = \dot{s}$$

$$\therefore \ v = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(\dot{x})^2 + (\dot{y})^2}$$





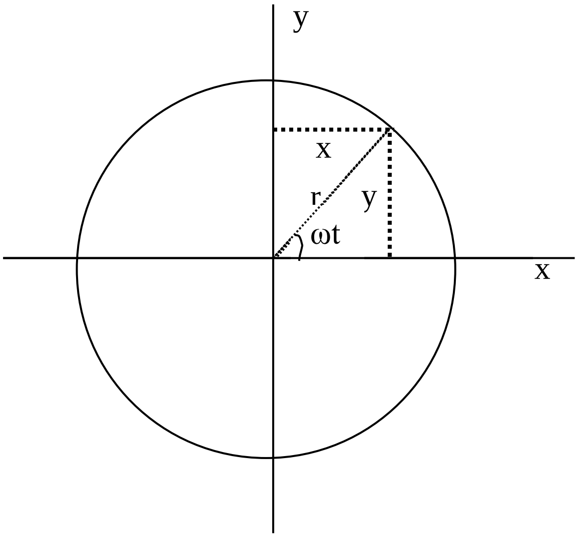
$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

$$a_x = \lim_{\Delta t \to \infty} \frac{\Delta \dot{x}}{\Delta t} = \frac{d\dot{x}}{dt} = \ddot{x}$$

$$a_y = \lim_{\Delta t \to \infty} \frac{\Delta \dot{y}}{\Delta t} = \frac{d\dot{y}}{dt} = \ddot{y}$$

$$\therefore \ a = \sqrt{(a_x)^2 + (a_y)^2} = \sqrt{(\ddot{x})^2 + (\ddot{y})^2}$$

- **Considering curvilinear motion of circular path:**



$x = r\cos\omega t \qquad \dot{x} = -r\omega\sin\omega t$

$y = r\sin\theta \qquad \dot{y} = r\,\omega\cos\omega t$

$x^2 + y^2 = r^2$

$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \omega r\sqrt{(\sin\omega t)^2 + (\cos\omega t)^2}$

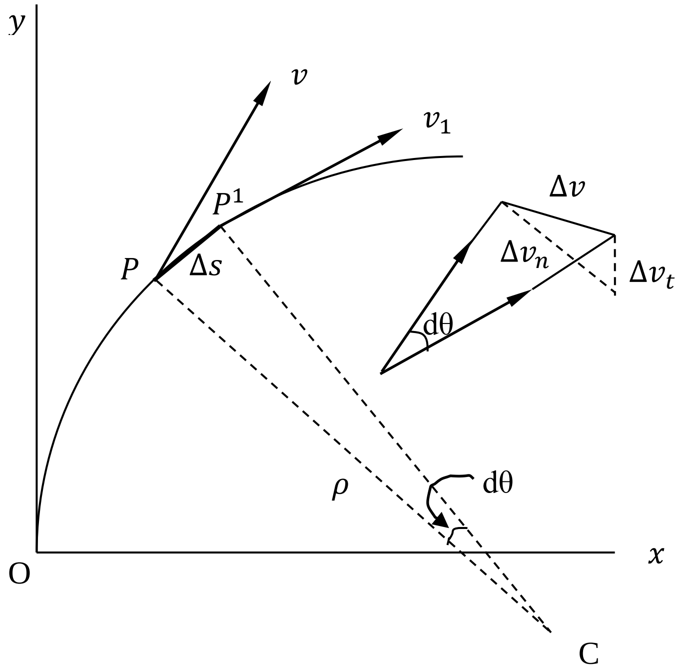
$v = \omega r$

$\ddot{x} = -r\omega^2\cos\omega t \qquad \ddot{y} = -r\,\omega^2\sin\omega t$

$a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \omega^2 r\sqrt{(\sin\omega t)^2 + (\cos\omega t)^2}$

$a = \omega^2 r = vr \quad since\ \therefore v = \omega r$

- **Normal and Tangential Accelerations:**



Let v is linear velocity at P

v_1 –linear velocity at P^1

ρ-radius f curvature of path at P

a_x = Acceleration in x-direction

a_y = Acceleration in x-direction

Consider another set of rectangular components normal acceleration $(a_n)$and tangential acceleration (a_t) at P

Let total change from v to v_1 . Δv_n and Δv_t be the velocity components parallel to normal and

$(a_t)_{avg} = \frac{\Delta v_t}{\Delta t} \qquad (a_n)_{avg} = \frac{\Delta v_n}{\Delta t}$

Let $\Delta\theta = \frac{\Delta s}{\rho}$; also from vector diagram,

$\Delta v_n \cong v\Delta\theta$ and $\Delta v_t \cong v_1 - v = \Delta v$

$\therefore (a_t) = \frac{\Delta v}{\Delta t}$ and $(a_n) = \frac{\Delta v_n}{\Delta t} \cong \frac{v\Delta\theta}{\Delta t} = \frac{v}{\rho}\frac{\Delta s}{\Delta t} = \frac{v^2}{\rho}$ but $v = \frac{ds}{dt} = \frac{\Delta s}{\Delta t}$

$$\boxed{\therefore a_t = \frac{\Delta v}{\Delta t}\ ;\ a_n = \frac{v^2}{\rho}}$$

Note-01:Tangential acceleration $\frac{\Delta v}{\Delta t}$ depends only on rate of change of speed while normal reaction, while normal acceleration always directed towards centre of curvature of path, depends upon nature of speed and curvature radius of path $\left(\frac{v^2}{\rho}\right)$. If the path is straight line curvature radius of path becomes zero $\left(\frac{1}{\rho} = 0\right)$.hence Normal acceleration (a_n) vanishes simply tangential acceleration exits $(a_t) = \frac{\Delta v}{\Delta t}$.

Note-02: if the speed of particle along curved path is constant, tangential acceleration $\frac{\Delta v}{\Delta t}$ vanishes and we have only normal acceleration $\left(\frac{v^2}{\rho}\right)$ always directed towards centre of curvature.

- **Differential equation of curvilinear path:**

In case of curvilinear motion, the acting force may vary in direction and magnitude and can be resolved both force and acceleration parallel to x and y axes

Force $X = \frac{\mathrm{W}}{g}\ddot{x}$ $Y = \frac{\mathrm{W}}{g}\ddot{y}$

If the acceleration and force are resolved normally and tangentially

Tangential force $F_t = \frac{\mathrm{W}}{g} a_t = \frac{\mathrm{W}}{g}\frac{\Delta v}{\Delta t}$

Similarly Normal force $F_n = \frac{\mathrm{W}}{g} a_n = \frac{\mathrm{W}}{g}\frac{v^2}{\rho}$ also known as centrifugal force

Ex: A particle of weight 'W' attached to a string of length 'l' whirls in a horizontal circular path with uniform speed 'υ' .Find the tensile force T in the string .

Sol: Since the speed is constant a_t is zero

There will be only normal acceleration a_n.

$a_n = \frac{v^2}{\rho}$ *in this case* $\rho = l$

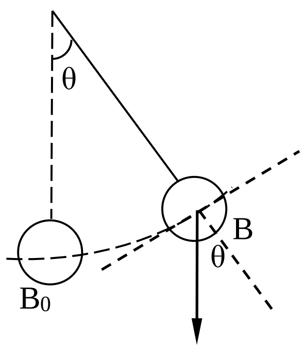
$\therefore a_n = \frac{v^2}{l}$

$\therefore Normal\ force\ T = N = \frac{\mathrm{W}.v^2}{g\ \rho}$

$\therefore N = \frac{\mathrm{W}.v^2}{g\ l}$

Ex: If the above simple pendulum of length 'l' has a bob of weight 'W' and hangs in vertical plane, find the time period T for small amplitude of swing in the plane in fig.

Sol: Consider a bob in position 'B' by a distance's' from 'B_o'.



The forces acting on bob are

i. Weight 'W'
ii. Tension in the string T

Projecting these forces on to the tangent

$s = \frac{\mathrm{W}}{g}.\frac{dv}{dt} = -\mathrm{W}\sin\theta \quad ---(1)$

Mass × acceleration = force

$$v = \frac{ds}{dt} \ and \ for \ small \ values \ of \ \theta, \theta = \frac{s}{l} \ and \ Sin\ \theta \cong \theta = \frac{s}{l}$$

Substituting in eq(1)

$$\frac{\mathrm{W}}{g} . \frac{d^2 s}{dt^2} = -\mathrm{W}\frac{s}{l}$$

$$\frac{d^2 s}{dt^2} + \frac{gs}{l} = 0$$

This is in the form of differential equation of simple harmonic motion (S. H. M)

$$\frac{d^2 s}{dt^2} + k.s = 0 \ or \ \frac{d^2 s}{dt^2} + p^2 s = 0$$

$$where\ k = \frac{g}{l}, p^2 = \frac{g}{l}$$

$$Time\ period\ (T) = 2\pi \sqrt{\frac{l}{g}}$$

Kinetics

Kinetics: The relation between kind of motion of a particle and the forces producing it is kinetics

Newton's Laws of motion:

- **First Newton's Laws**
  Everybody continuous in the state of rest or of uniform motion in straight line unless otherwise it may be compelled by a force to change the state. This is also called "**Inertia law**".
- **Second Newton's Laws:**
  The acceleration of a given particle is proportional to the force applied to it and takes place in the direction of straight line in which force acts. This is also known as "**momentum law**". This law gives the mathematical expression of a force

  Momentum: It is quantity of motion possessed by a body

  Momentum = Mass x velocity

  Units of momentum: kg-m/sec

  Initial Momentum = mv_0

  Final Momentum =mv

  Change in momentum = $mv - mv_0$

  Rate of change of momentum =$m\frac{(v - v_0)}{t} = ma$ x

According to the Newton second law rate of change of momentum is directly proportional to the applied force:

$$F \propto ma$$

$F = Kma$ Where K is constant of proportionality of $a = 1m/sec^2$ and $m = 1kg \ and \ k = 1$

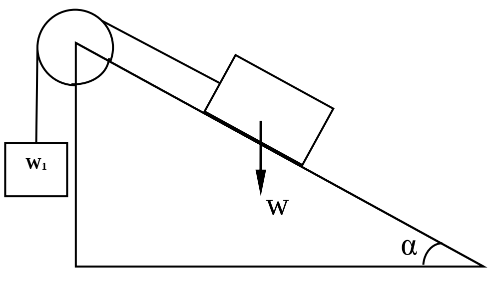
$$F = ma = 1kg\frac{m}{sec^2} = 1Newton$$

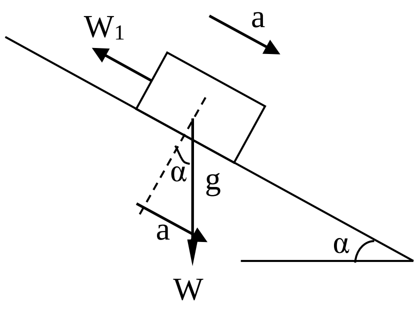
By varying 'α', the angle of inclined plane, a relation can be obtained between acceleration along inclined plane and acceleration due to gravity.

$\sin\alpha = \frac{a}{g}$

$\therefore a = g \sin\alpha$

$when\ \alpha = \frac{\pi}{2}, a = g$

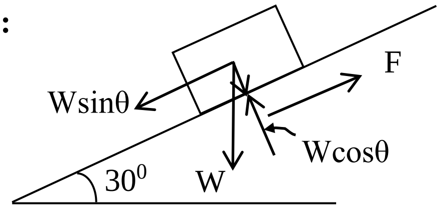




Also $W_1 = W \sin\alpha$

Ex: A small block of weight W rest on inclined plane as shown in fig .Friction is such that sliding of the block impends when $\alpha=30^0$ what the acceleration of the block will have when $\alpha=45^0$ .Neglect difference of static and kinematic friction

Sol:



$R = W\cos\theta$ and $F = \mu R = \mu W\cos\theta$

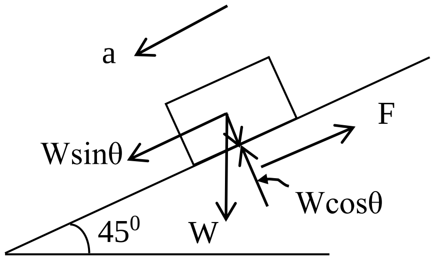
$F = W\sin\theta$

$\mu R = W\sin\theta$

$\mu W\cos\theta = W\sin\theta$

$$\mu = \frac{\sin\theta}{\cos\theta} = \tan\theta = \tan 30^0$$

$\mu = 0.577$



$R = W\cos 45^0$ and $F = \mu R = 0.577\, W\cos 45^0$

Net force= $W\sin 45^0 - 0.577\, W\cos 45^0$

Net force= $\frac{W}{g} a$

$W\sin 45^0 - 0.577\, W\cos 45^0 = \frac{W}{g} a$

$\sin 45^0 - 0.577\cos 45^0 = \frac{1}{g} a$

$a = 0.3g$

- **Third Law:** For every action there is an equal and opposite reaction or Mutual actions of any two bodies are always equal and opposite directions.
  From Galileo's experiments, the gravity force W produces and acceleration of particle equal to 'g'

$$W = mg$$
$$\frac{F}{W} = \frac{ma}{mg} = \frac{a}{g}$$

Absolute and gravitational units:

"Newton" is absolute unit of a force and "Kg" is gravitational unit

$$\therefore 1kg = g\ newtons = 9.81\ newtons$$

Differential equation of rectilinear motion: Taking motion of a particle in x-direction, and using the acceleration as $\ddot{x}$, the resultant acting force in x-direction

$$X = \frac{W}{g}\ddot{x} --- \text{Differential Equation}$$

D'Alebert's principle:

The differential equation of rectilinear motion of a particle is $X = m\ddot{x}$

It can be written in the form $X - m\ddot{x} = 0$ --- (1)

Where X – Resultant force in x – direction of all applied forces

M – Mass of the particle

Equation (1) is of the same form as an equation of static equilibrium and may be stated as equation of dynamic equilibrium.

In Equation (1) ($- m\ddot{x}$) is the inertia force equal to the product of mass of the particle and its acceleration and directed oppositely to the acceleration of the system.

If W – total weight of the body

For a rectilinear motion of a rigid body, the equation of dynamic equilibrium is,

$$\sum X_i + \left(-\frac{W}{g}.\ddot{x}\right) = 0 \quad --- (2)$$

D'Alembert principle is that the equation of motion could be written as equilibrium equations simply by introducing inertia forces in addition to the real forces acting on a system.

Ex: Assume $m_1 > m_2$

Motion of m_1 is downwards and motion of m_2 is upwards

Corresponding inertia force act in opposite direction $m_1\ddot{x}$ upward and $m_2\ddot{x}$ downward.

Adding these inertia forces to the real forces we obtain a system of forces in equilibrium.

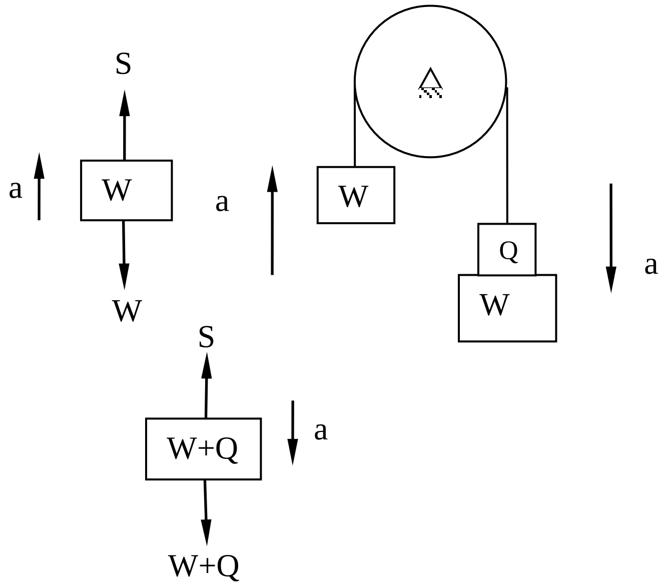
Considering the system as a whole,

$$w_2 + m_2\ddot{x} - S = w_1 - m_1\ddot{x} - S$$

$$\ddot{x}\left(\frac{w_1 + w_2}{g}\right) = w_1 - w_2$$

$$\ddot{x} = \frac{(w_1 - w_2)g}{(w_1 + w_2)}$$

Ex: Two equal weights W each and a single weight Q are attached to the ends of a flexible but inextensible cord overhanging of pulley as shown in fig. If the system moves with constant acceleration ‘a’ as indicated by the arrows, find the magnitude of weight ‘Q’. Neglect the air resistance and inertia of the pulley.



Sol: Considering equilibrium of left weight W,

$$S - W = \frac{W\,a}{g} \quad ---(1)$$

Considering equilibrium of right weight,

$$(W + Q) - S = \frac{W+Q}{g}.a \quad ---(2)$$

From (1) $S = W + \frac{W\,a}{g}$

Substituting in (2)

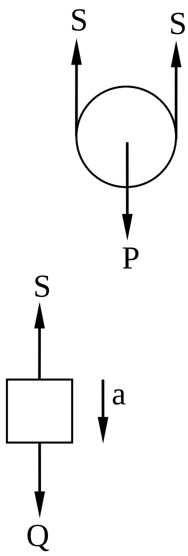
$W + Q - W - \frac{W\,a}{g} = \frac{W\,a}{g} + \frac{Q\,a}{g}$

$Q\left(1 - \frac{a}{g}\right) = \frac{2\,W\,a}{g}$

$\therefore Q\left(\frac{g-a}{g}\right) = \frac{2\,W\,a}{g}$

$\therefore Q = \frac{2\,W\,a}{(g-a)}$

Ex: Two weights P and Q are connected by the arrangements shown in Fig. Neglecting friction and inertia of pulleys and cord, find the acceleration ‘a’ of the weight Q.Assume P=20kg and Q=15kg



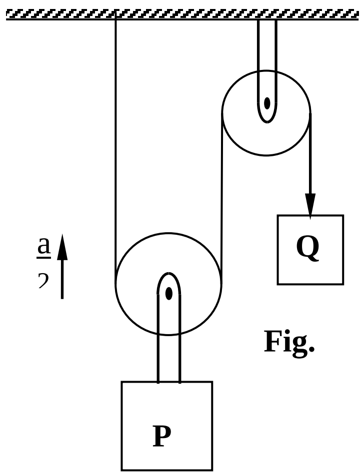


Fig.

Sol: Consider FBD of ‘P’

Consider FBD of pulley A,

$2S - P = \frac{P}{g}\left(\frac{a}{2}\right)$

$2S - P = \frac{Pa}{2g} \quad ---(1)$

$2S = P + \frac{Pa}{2g}$

$S = \frac{P}{2}\left[1 + \frac{a}{2g}\right]$

Considering FBD of ‘Q’

$Q - S = \frac{Qa}{g} \quad ---(2)$

Substituting ‘S’

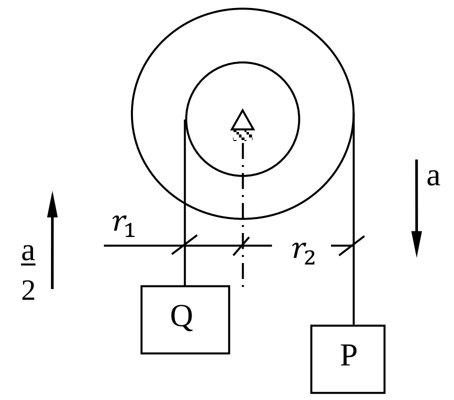
$Q - \frac{P}{2}\left(1 + \frac{a}{2g}\right) = \frac{Qa}{g}$

$Q - \frac{P}{2} - \frac{Pa}{4g} = \frac{Qa}{g}$

$\frac{a}{g}\left(\frac{P}{4} + Q\right) = Q - \frac{P}{2}$

$$\therefore a = \frac{g\left(Q - \frac{P}{2}\right)}{\left(\frac{P}{4} + Q\right)} = \frac{9.81\left(15 - \frac{20}{2}\right)}{\left(\frac{20}{4} + 15\right)} = \frac{9.81(5)}{20} = \frac{9.81}{4} = 2.45\ m/sec^2$$

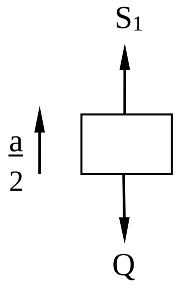
Ex: Neglecting friction and inertia of the two step pulley as shown in fig, find the acceleration 'a' of the following weight 'P'. Assume P= 4 kg, Q= 6 kg and $r_2 = 2r_1$



Sol: In step pulley by the time 'P' makes one revolution through perimeter $2\pi r_2$, 'Q' will also make a revolution of $2\pi r_1$.

$$\therefore \frac{a_1}{a} = \frac{2\pi r_1}{2\pi r_2} = \frac{r_1}{2r_1} = \frac{1}{2}$$

$$\therefore a_1 = \frac{1}{2}\, a \rightarrow accelration\ of\ Q$$



Considering the resultant force of Q

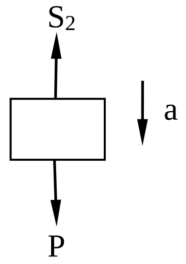
$$S_1 - Q = \frac{Q}{g}\frac{a}{2} \quad --- (1)$$

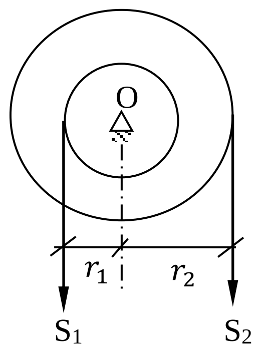
Considering the resultant force of P

$$P - S_2 = \frac{Pa}{g} \quad --- (2)$$

From eqn (1)

$$S_1 = Q + \frac{Qa}{2g}$$





FBD of pulley system

Also from FBD of pulley system

Taking moments about 'O',

$S_1 \times r_1 = S_2 \times r_2$

$S_1 \times r_1 = S_2 \times 2\, r_1$

$S_1 = 2\, S_2$ --- (3)

Substituting in (1)

$$2S_2 - Q = \frac{Q\, a}{2\, g}$$

Substituting S_2 from (2)

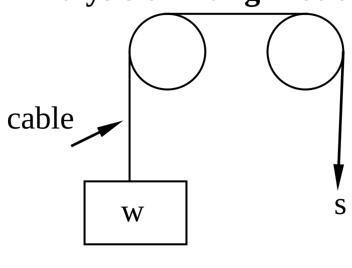
$$2P - \frac{2\, P\, a}{g} - Q = \frac{Q\, a}{2\, g}$$

$$\frac{2\, P\, a}{g} + \frac{Q\, a}{2\, g} = 2\, P - Q$$

$$\frac{a}{g}\left[2P + \frac{Q}{2}\right] = 2\, P - Q$$

$$\therefore a = \frac{g(2\, P - Q)}{2P + \frac{Q}{2}} = \frac{g(8-6)}{8+3} = \frac{2}{11}g = \frac{2}{11} \times 9.81 = 1.784\ m/s^2$$

Analysis of lifting Motion:



Elevator

Let W - Weight carried by the lift

m - Mass carried by the lift

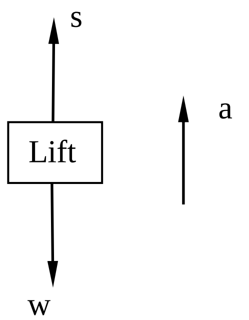
$$\therefore m = \frac{W}{g}$$

$a - uniform\ acceleration\ of\ lift$

S-Tension in cable of lift

The lift may be travel upwards or downwards

Case (i) : when the lift moves upwards



Since lift is moving upwards acceleration is upwards and hence $s > w$

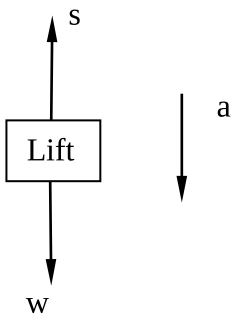
Resultant force = s - w

As per the Newton's second law,

Net force = $ma = \frac{w}{g}a$

$$s - w = \frac{w}{g}a \qquad s = w + \frac{w}{g}a = w\left(\frac{g+a}{g}\right)$$

Case (ii) :when the lift moves downwards



Since lift is moving downwards acceleration is downwards and hence

$w > s$

Resultant force = $w - s$

As per the Newton's second law,

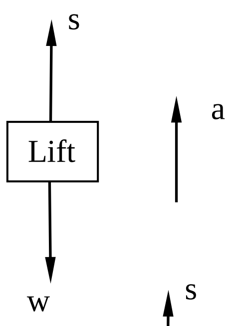
Net force = $ma = \frac{w}{g}a$

$$w - s = \frac{w}{g}a \qquad s = w - \frac{w}{g}a = w\left(\frac{g-a}{g}\right)$$

Ex: A lift has as upward acceleration of 1.225 m/sec^2

i. What pressure will a man weighting 500N exert on the floor of the lift
ii. What pressure would he exert if the lift had an acceleration of 1.225 m/sec^2 downwards
iii. What upward acceleration would cause his weight to exert a pressure of 600N on the floor? Assume $g = 9.81\ m/sec^2$.

Sol: 1. <u>When lift is in upward motion:</u>

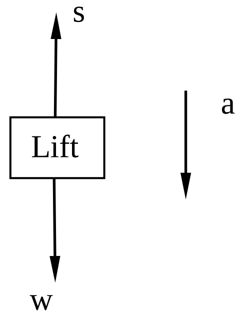


S > W

$\therefore S - W = \frac{W}{g}a$

$S = \frac{W}{g}(g + a) = \frac{500}{9.8}(9.8 + 1.225) = 562.5\ N$

2. <u>When lift is in downward motion:</u>



W > S

$\therefore W - S = \frac{W}{g}a$

$S = \frac{W}{g}(g - a) = \frac{500}{9.8}(9.8 - 1.225) = 437.5\ N$

3. <u>When S = 600N and W = 500N lift is in upward motion</u>

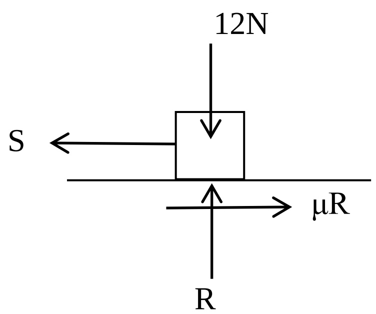
$S - W = \frac{W}{g}a$

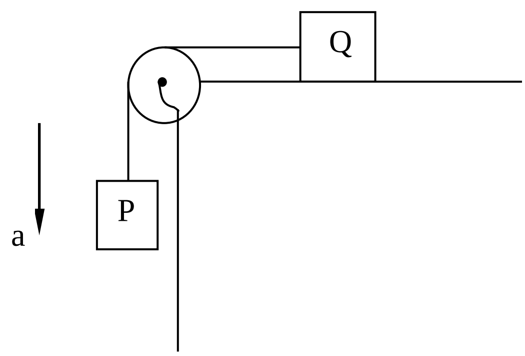
$600 - 500 = \frac{500}{9.8}a$

$a = \frac{100\times 9.8}{500} = 1.96 m/s^2.$

Ex: Referring to the fig, find the acceleration of a falling weight P, if the coefficient of friction between the block Q and the horizontal plane on which it slides is μ .Neglect the inertia of the pulley and friction. Assume P=10kg, Q=12kg and μ=1/3.

Sol: From Free body Diagram of Block Q:

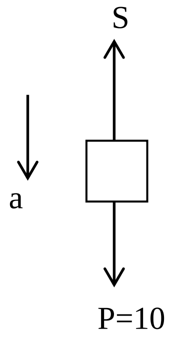




$S - \mu R = \frac{W}{g} a$

$S - \frac{12}{3} = \frac{12}{9.81} a - - - (1)$

From Free body Diagram of Block P:



$P - S = \frac{W}{g} a$

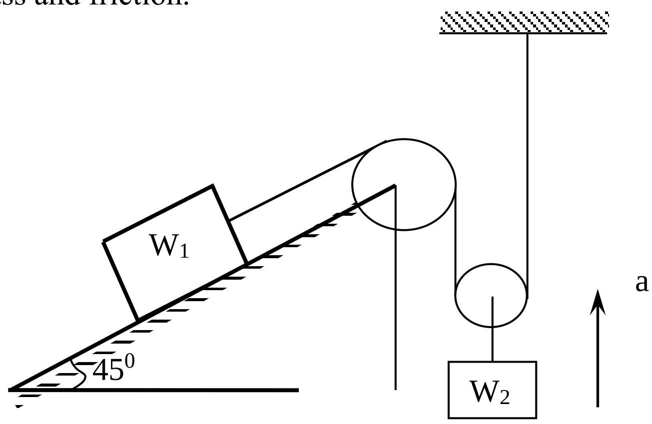
$S = 10 - \frac{10}{g} a \quad - - - (2)$

Substituting in Eq-(1)

$10 - \frac{10}{g} a - 4 = \frac{12}{g} a$

$6 = \frac{22}{g} a \qquad \therefore a = \frac{3}{11} g$

Ex: Find the tension S in the string during the motion of the system shown in fig .If W_1 =200kg and W_2 =100kg.the system is in the vertical plane and coefficient of friction between the inclined plane and block W_1 is μ =0.2. Assume the pulley is without mass and friction.



Sol:

$R = 200cos45^0 = 141.4kg$

F = μ R

$F = 0.2 \times 141.4 = 28.28\ kg$

Resolving along the plane

$200sin45^0 - F - S = \frac{200}{g}(2a)$

$141.4 - 28.28 - S = \frac{200}{9.81}(2a)$

$113.12 - S = 40.77a$

$S = 113.12 - 40.77a - - - (1)$

From FBD of W_2:

$2S - 100 = \frac{100}{g} a$

Substituting 'S' value

$2(113.12 - 40.77a) - 100 = \frac{100}{9.81} a$

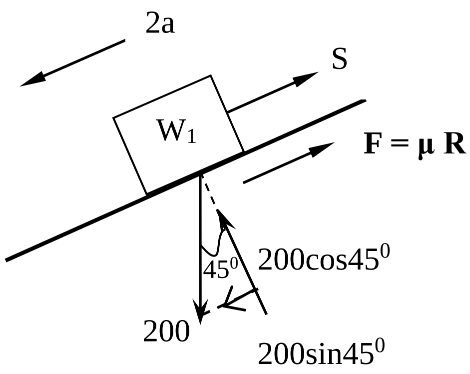
$226.24 - 81.54a - 100 = 10.19a$

$91.73a = 126.24$

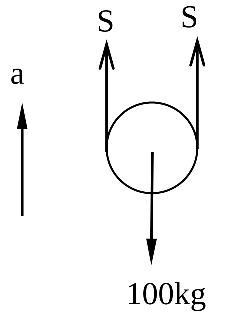
$a = \frac{126.24}{91.73} = 1.376\ {}^{m}/_{sec^2}$

$S = 113.12 - 40.77 \times 1.376$

$S = 113.12 - 56.1 = 57.02kg$



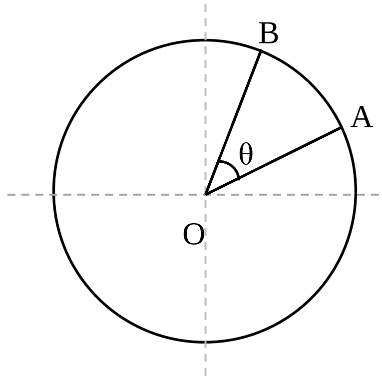
FBD of block W_1



FBD of block W_2

Rotation about fixed axis:

Angular Motion:



Let θ be the angular displacement in radians

Angular velocity $\omega = \frac{d\theta}{dt}$ ---------------------------- (1)

Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$;

$\alpha = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega \frac{d\omega}{d\theta} \quad - - - (2)$

$s = r\theta\ ;\ ds = rd\theta$

$\omega = \dfrac{\theta_2 - \theta_1}{t_2 - t_1}$

$v = \dfrac{ds}{dt} = \dfrac{d\theta}{dt} = r\omega$

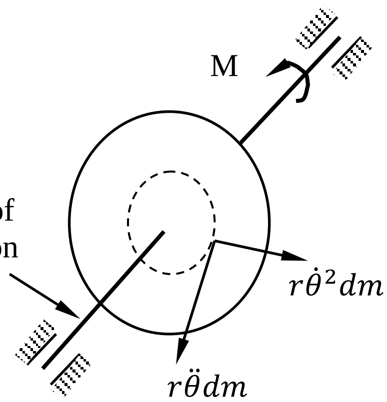
Tangential acceleration $\alpha_t = \frac{dv}{dt} = r\frac{d^2\theta}{dt^2} = r\ddot{\theta}$

Normal acceleration $\alpha_n = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2 = r\dot{\theta}^2$

$\alpha_t = a_t = \dfrac{dv}{dt} = r\dfrac{d^2\theta}{dt^2} = r\ddot{\theta}$

Here $\theta = \omega t$ $\omega = \frac{2\pi N}{60}$ $radians$

- **Equation of motion for a rigid body rotating about a fixed axis:**



Consider the body as a system of rigidly connected particles and apply inertia force to each particle.
Inertia force together with external forces contributes forces system in dynamic Equilibrium. Internal forces (actions and reaction forces) balance each other.

To calculate moment of inertia forces about the axis of rotation, tangential components to be considered and moment of radial components becomes zero.

Tangential component of inertia force = $a_t dm = r\ddot{\theta} dm$

Here a_t - Tangential acceleration

$\ddot{\theta}$-Angular acceleration

Sum of all the moments of inertia forces =Resultant moment of all external forces

Let M be the resultant moment of all the external forces

$\therefore -\int r^2\ddot{\theta}dm + M = 0$ -------------------------------- (1)

We know that, mass moment of inertia about the axis of rotation $I = \int r^2 dm$, then Eq (1) becomes.

$$\boxed{I\ddot{\theta} = M}$$

This the equation of motion of a rigid body about a fixed axis, this similar to equation $X = \frac{W}{g}\ddot{x}$

Ex-01:
The rotation of a fly wheel is governed by the equation $\omega = 3t^2 - 2t + 2$ where ω is radian per second and 't' is in seconds after one second from start the angular displacement was 4 radians .Determine the angular displacement ,angular velocity and angular acceleration of the flywheel when t = 3 sec.

Sol: $\omega = 3t^2 - 2t + 2$

$\omega = \frac{d\theta}{dt} = 3t^2 - 2t + 2$

Integrating

$\theta = t^3 - t^2 + 2t + C$

When t = 1, θ = 4

4 = 1 - 1 + 2 + C

C = 2

Equation becomes, $\theta = t^3 - t^2 + 2t + 2$

When t = 3 sec

θ = 27 – 9 + 6 + 2 = 26 radians

$$\omega = \frac{d\theta}{dt} = 3t^2 - 2t + 2 = 27 - 6 + 2 = 23 \text{ rad/sec}$$

$$angular\ acceleration = \frac{d^2\theta}{dt^2} = \ddot{\theta} = 6t - 2 = 18 - 2 = 16\, rad/sec^2$$

Ex:A flywheel of M.I, I=700 kg-sec^2-cm with respect to its axis of rotation and making 100 rpm ,if left alone comes to rest with constant angular deceleration in 52 seconds, owing to friction in bearings .Determine the friction couple that produces this angular deceleration.

Sol: Given I = 700 kg-sec^2.cm, t = 52 sec

$$\ddot{\theta} = \frac{\dot{\theta}_0 - \dot{\theta}}{t}\ ;\ \dot{\theta} = \frac{100.\ 2\pi}{60} = \frac{10\,\pi}{3}\ rad/sec$$

$$\therefore\ \ddot{\theta} = \frac{\dot{\theta}_0 - \dot{\theta}}{t} = \frac{10\,\pi}{3\times 52} = 0.201\, rad/sec^2$$

$$M = I\ddot{\theta} = 700\ \times 0.201 = 140.7\ kg - cm$$

Ex: Determine Equivalent mass moment of inertia shown in figure below.

Sol: Tangential acceleration = rθ̈

$$\therefore\ inertia\ force = \frac{w}{g}(r\ddot{\theta})$$

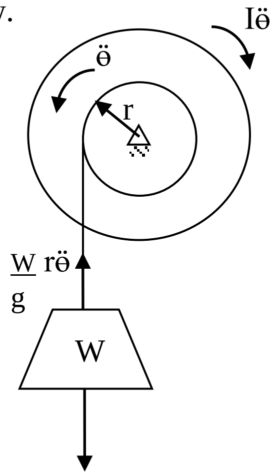
Inertia couple = I θ̈

Taking moments about ‘O’,

$$Wr - \frac{w}{g}.r^2\ddot{\theta} = I.\ddot{\theta} + \frac{w}{g}(r\ddot{\theta})$$

$$\ddot{\theta}\ \left[I + \frac{W}{g}.r^2\right] = W.r$$

$$where\ I_0 = I + \frac{W}{g}\ r^2\ is\ equivilent\ mass\ moment\ of\ inertia$$



UNIT –VI

Syllabus:

Work – Energy Method

Definition of work, power and energy – Potential energy and kinetic energy – Principle of conservation of energy – Application to particle motion and connected systems – Work done by spring.

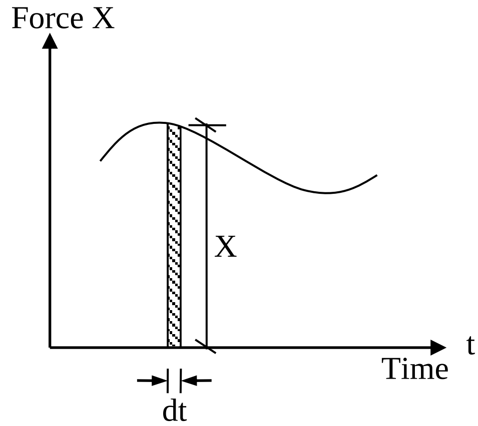
Impulse and Momentum Method

Definition of impulse and Momentum – Law of conservation of momentum – Simple problems; Collision of bodies - Direct impact and plastic impact, Elastic and Semi elastic impacts – Coefficient of restitution – Simple problems.

Learning Material

Momentum and impulse:

Differential equation of rectilinear motion of a particle is



$$\frac{W}{g}.\frac{d^2x}{dt^2} = X$$

$$\frac{d}{dt}\left(\frac{W}{g}.\frac{dx}{dt}\right) = X$$

$$d\left(\frac{W}{g}.\frac{dx}{dt}\right) = X.dt \quad --- (1)$$

Where X – force is function of time

Impulse is the product of force and time. Unit of impulse is N – sec.

Momentum is the product of mass and velocity.

Unit of momentum = $\frac{kg.m}{sec}$ $or\ N - sec$. Since 1N $= 1\frac{kg.m}{sec^2}$

Relation between momentum and impulse:

Differential change of momentum of a particle during an element time 'dt' is equal to the impulse of acting force during that time.

∴ *Rate of change of momentum is impulse.*

Integrating eqn(1)

$$\frac{W}{g}.\dot{x} + C = \int_0^t X.dt$$

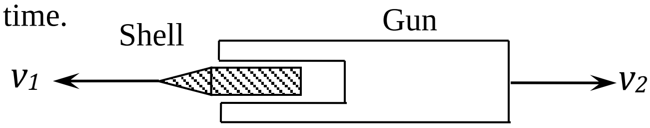
When t = 0, $\dot{x} = \dot{x}_0$

$$\therefore C = -\frac{W}{g}.\dot{x}_0$$

$$\frac{W}{g}.\dot{x} - \frac{W}{g}.\dot{x}_0 = \int_0^t X.dt$$

∴ Total change of momentum is impulse during the same time.

Ex: Gun and shell problem



Let W_1 – weight of shell, W_2 – weight of gun

Applying Momentum – Impulse Principle,

For Shell:

$$\frac{W_1}{g}.v_1 - \frac{W_1}{g}.v_0 = F\int dt \qquad ---(1)$$

For Gun:

$$\frac{W_2}{g}.v_2 - \frac{W_2}{g}.v_0 = F\int dt \qquad ---(2)$$

Assuming initial velocities of both gun and shell are zero,

$$\therefore \frac{W_1}{g}.v_1 = F\int dt = \frac{W_2}{g}.v_2$$

$$\frac{v_1}{v_2} = \frac{W_2}{W_1}$$

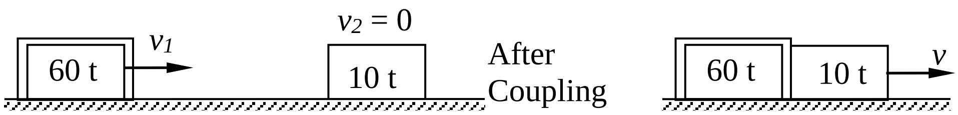
Law of conservation of momentum:

In case of any system of particles for which no external forces are applied, the total momentum of the system remains unchanged since the total impulse is zero.

This is the *principle of law of conservation of momentum*.

Ex: A locomotive weighing 60 tonnes has a velocity of 16 kmph and backs into a fright car weighing 10 tonnes which is at rest on a level track. After coupling is made, with what velocity 'v' will the entire system continue to move?

Sol:



Applying law of conservation of momentum

$$\frac{60}{g}.v_1 + \frac{10}{g}.v_2 = \frac{60+10}{g}.v$$

$$\therefore \frac{60\times16}{g} = \frac{70}{g}.v$$

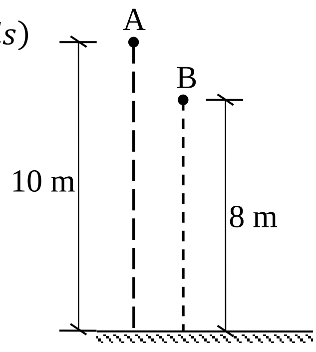
$$\therefore v = \frac{60\times16}{70} = 13.71\ kmph$$

Ex: A glass marble, whose weight is 0.2 N, falls from a height of 10 m and rebounds to a height of 8 m. find the impulse and average force between the marble and the floor, if the time during which they are in contact is 0.1 second.

Ans: Applying kinematics of freely falling body

The velocity with which the ball strikes the ground $v_1 = \sqrt{2gh_1}$

$$v_1 = \sqrt{2g\times10} = \sqrt{2\times9.81\times10} = 14\ m/s \quad (downwards)$$



Similarly applying kinematic equation for marble moving upwards

$$v_2 = \sqrt{2g\times h_2} = \sqrt{2\times9.81\times8} = 12.53\ m/s$$

Taking upward as positive and downward as negative,

Applying momentum equation,

$$Impulse = \frac{\mathrm{W}}{g}(v_2 - v_1) = \frac{0.2}{9.81}(12.53 - (-14.0))$$

$$= \frac{0.2}{9.81}\times 26.53 = 0.541\ N - sec$$

If F is the average force between marble and the floor,

F. t = 0.541

$$F = \frac{0.541}{0.1} = 5.41\ N$$

Ex: A block of weight 130 N is on an inclined plane, whose slope is 5 vertical to 12 horizontal. Its initial velocity down the plane is 2.4 m/s. What will be its velocity 5 seconds later? Take μ = 0.3

Sol: $Tan\ \theta = \frac{5}{12}$

θ = $22^0.62$

N = W Cos θ = 130 Cos $22^0.62$ = 120 N

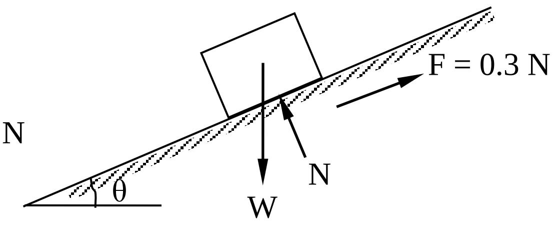
F = μ N = 0.3 × 120 = 36 N

Forces down the plane = W sin θ – F

= 130 Sin $22^0.62$ – 36 = 14 N

Initial velocity = 2.4 m/s = v_0

Let final Velocity = v



Time interval, t = 5 sec

Applying impulse – Momentum Equation

$$F.t = \frac{W}{g}(v - v_0)$$

$$14 \times 5 = \frac{130}{9.81}(v - 2.4)$$

$$(v - 2.4) = 5.28$$

$$v = 5.28 + 2.4 = 7.68\ m/s$$

Impact

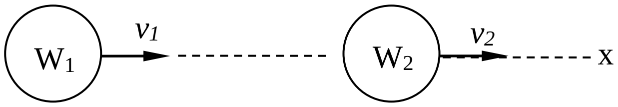
Collision of two moving bodies with very large magnitude of active and reactive forces during a very short interval of time is called "***Impact***".

The magnitude of force and direction of impact depends upon the shape, velocity and elastic properties of the bodies.

Types of impact:

a) Direct central impact
b) Plastic impact
c) Elastic impact
d) Semi – elastic impact

a) Direct Central Impact:



Let before impact, balls are moving with velocities v_1 & v_2

Let before impact $v_1 > v_2$

During impact, two equal and opposite forces, action and reaction are produced at the point of contact.

Let v_1' and v_2' – velocities of balls after impact

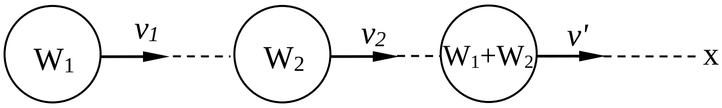
As per law of conservation of momentum,

Total momentum before impact = total momentum after impact

$$\frac{W_1 v_1}{g} + \frac{W_2 v_2}{g} = \frac{W_1 v_1'}{g} + \frac{W_2 v_2'}{g} \quad --- (1)$$

b) Plastic impact:

In this case, the velocity of the striking ball I gradually diminishes owing to the reaction from ball II and at the same time, the velocity of ball II increases owing to the action of ball I.



Let v' – common velocity of (W_1+W_2) after impact

Total momentum before impact = Total momentum after impact

$$\frac{W_1 v_1}{g} + \frac{W_2 v_2}{g} = \left(\frac{W_1}{g} + \frac{W_2}{g}\right) v'$$

or

$$v' = \frac{W_1 v_1 + W_2 v_2}{W_1 + W_2} \quad --- (2)$$

c) Elastic Impact:

Assuming the material of sphere is perfectly elastic,

Ex: Hardened polished steel balls or glass balls

In case of perfect elasticity, there must be no loss in the energy of the system and in addition momentum equation is also available.

Using energy equation before and after impact

$$\frac{W_1 {v_1}^2}{2g} + \frac{W_2 {v_2}^2}{2g} = \frac{W_1 {v_1}'^2}{2g} + \frac{W_2 {v_2}'^2}{2g} \quad --- (1)$$

Where v_1 and v_2 are velocities before impact and v_1' and v_2' are velocities after impact

Momentum equation is

$$\frac{W_1 v_1}{g} + \frac{W_2 v_2}{g} = \frac{W_1 v_1{'}}{g} + \frac{W_2 v_2{'}}{g} \quad ---(2)$$

From Equation (2)

$$W_1 (v_1 - v_1') = W_2 (v_2' - v_2) \qquad ----- (3)$$

From Equation (1)

$$W_1 (v_1^2 - v_1'^2) = W_2 (v_2'^2 - v_2^2) \qquad ----- (4)$$

Dividing (4) with (3)

$$\frac{W_1(v_1{}^2 - v_1{'}^2)}{W_1(v_1 - v_1')} = \frac{W_2(v_2{'}^2 - v_2{}^2)}{W_2(v_2' - v_2)}$$

$$v_1 + v_1' = v_2' + v_2$$

or $$v_1' - v_2' = -(v_1 - v_2) \qquad ----- (5)$$

This states that for an elastic impact, the relative velocity after impact has the same magnitude as that before impact but in opposite direction.

d) Semi – Elastic Impact:

Under actual conditions, some deviation is expected from perfect elasticity and owing to this fact there always will be some loss in energy of the system during impact so that the relative velocity after impact is smaller than before.

Therefore equation (5) becomes

$$v_1' - v_2' = -e\,(v_1 - v_2) \qquad ----- (6)$$

Where 'e' is called "**coefficient of restitution**" which is always less than unity.

$$\therefore e = \frac{-(v_1' - v_2{'})}{(v_1 - v_2)}$$

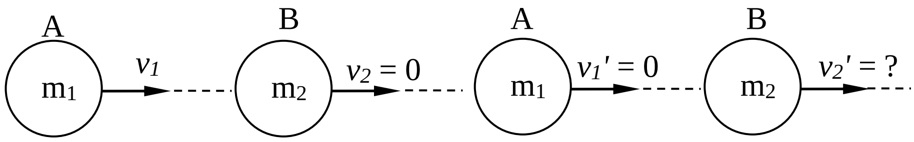
Ex: Ball A of mass 1 kg moving with a velocity of 2 m/s strikes directly and ball B of mass 2 kg at rest. The ball A after striking, comes to rest, Find the velocity of ball B after striking and the coefficient of restitution.

Sol: Problem is on direct impact

$m_1 = 1$ kg

$v_1 = 2$ m/s

$m_2 = 2$ kg



Before Impact After Impact

Total momentum before impact = Total momentum after impact

$$\frac{W_1 v_1}{g} + \frac{W_2 v_2}{g} = \frac{W_1 v_1{'}}{g} + \frac{W_2 v_2{'}}{g}$$

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$1 \times 2 + 0 = 0 + 2 v_2'$$

$$v_2' = 1 \text{ m/s}$$

Coefficient of restitution,

$$e = \frac{-(v_1' - v_2')}{(v_1 - v_2)} \qquad since\ v_1' and\ v_2 = 0,$$

$$= \frac{-(0-1)}{(2-0)} = +\frac{1}{2}$$

Ex: A glass marble drops from a height of 3m upon a horizontal floor. If the coefficient of restitution is 0.9, find the height to which it rises after impact?

Sol: h = 3m, e = 0.9, h_1 = ?

Velocity of floor before and after impact, v_2 & $v_2' = 0$

$$e = \frac{-(v_1' - v_2')}{(v_1 - v_2)} = \frac{-\left(\sqrt{2gh_1} - 0\right)}{\sqrt{2gh} - 0} = 0.9$$

Squaring on both sides

$$\frac{\cancel{2g}h_1}{\cancel{2g}h} = 0.81$$

$$\frac{h_1}{3} = 0.81$$

$$\therefore h_1 = 3 \times 0.81 = 2.43\ m$$

Work and Energy

Equation of rectilinear motion of a particle

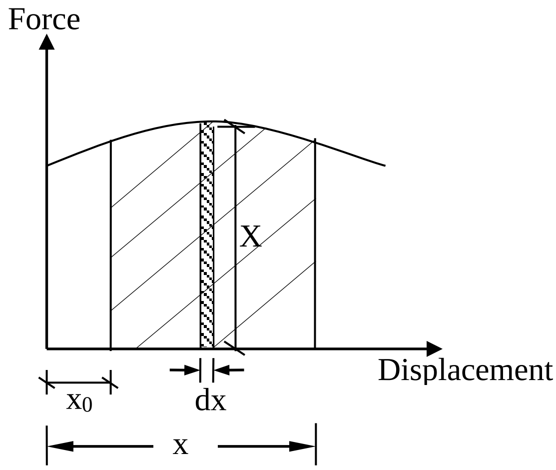
$$X = \frac{W}{g}.\frac{d^2x}{dt^2} = \frac{W}{g}.\frac{d\,\dot{x}}{dt}$$

$$\frac{W}{g}.\frac{d\,\dot{x}}{dt} = X$$

Multiplying both sides by dx

$$\frac{W}{g}.\frac{dx}{dt}.d\dot{x} = X.dx$$

$$d\left[\frac{W}{g}.\frac{\dot{x}^2}{2}\right] = X.dx \quad ---(1)$$



Where X – Force with function of displacement

Work done by force X on infinitesimal displacement dx = X. dx,

$$and\ \ K.E = \frac{W}{g}.\frac{\dot{x}^2}{2}$$

Therefore from Equation (1)

Differential change in K.E is equal to the work done by the active force.

Energy is internal property and work done is external property.

Integrating Equation (1)

$$\frac{W}{g}.\frac{\dot{x}^2}{2} + c = \int_0^x X.dx$$

When x = x_o and $\dot{x} = \dot{x}_o$

$$c = -\frac{W}{g}.\frac{{\dot{x}_0}^2}{2} + \int_0^{x_0} X.dx$$

Substituting 'c' in above equation

$$\frac{W}{g}.\frac{\dot{x}^2}{2} - \frac{W}{g}.\frac{{\dot{x}_0}^2}{2} = \int_{x_0}^{x} X.dx \quad ---(2)$$

This is represented by area of force – displacement diagram.

This proves that change in K.E is equal to Work done.

Types of energy:

1. **<u>Potential Energy</u>**: It is the energy of a body by virtue of its position.

   U = m g h = W h

   Ex: a compressed string has P.E because it can do work in recovering its original shape.
2. **<u>Kinetic Energy</u>**: Energy of a body possessed by virtue of its velocity or motion is known as kinetic energy.

   Ex: A body falling from a height 'h' and strikes ground

   When it starts from rest $v_o = 0$

   Change in K.E = Work done

$$\frac{Wv^2}{2g} - \frac{W{v_0}^2}{2g} = W.h$$

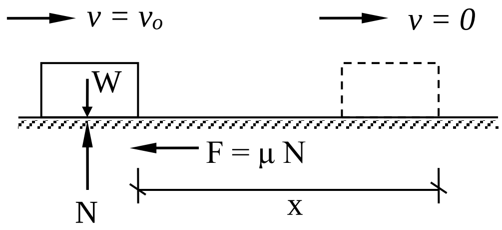
$$Since\ v_0 = 0,\ \ \frac{\cancel{W}v^2}{2g} = \cancel{W}.h$$
$$\therefore v = \sqrt{2gh}$$

Ex: If a weight 'W' is given an initial velocity 'v_o' along a rough horizontal plane and comes to rest by friction in a distance 'x'. Determine the coefficient of friction that is independent of velocity.

Sol: Frictional force F = μ N

N = W

Therefore F = μ W --- (1)



Since force is opposite to the direction of displacement, Work done is negative.

Work done = - μ W. x

Change in K.E = work done

$$\frac{Wv^2}{2g} - \frac{W{v_0}^2}{2g} = work\ done$$

$$0 - \frac{W{v_0}^2}{2g} = -\mu\, Wx$$

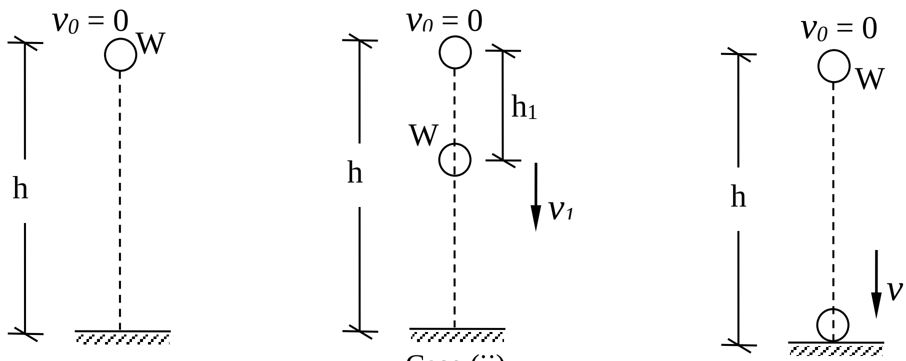
$$\mu = \frac{{v_0}^2}{2gx}$$

∴ μ is proportional to the square of velocity and displacement.

Law of conservation of energy:

Statement: Energy can neither be created nor be destroyed; but it can be transformed from one form to other.

Total energy possessed by a body remains unchanged.



Case (i) Case (ii) Case (iii)

Ex: ***Case 1***: Let a body of weight 'W' is initially at rest at a height 'h' above ground surface.

Then P.E = W.h

K.E = 0 (no motion)

Therefore Total energy = P.E + K.E = Wh + 0 = Wh --- (1)

Case 2: Let W is falling through a height h_1 and the velocity at this stage is v_1, then

Then change in K.E = $\frac{W{v_1}^2}{2g} - \frac{W{v_0}^2}{2g} = \frac{W{v_1}^2}{2g}$

P.E at this stage = W (h – h_1)

Total energy = $\frac{W{v_1}^2}{2g} + W(h - h_1)$

But we know that $v_1^2 - v_0^2 = 2gh_1$

Therefore $v_1^2 = 2gh_1$

$\therefore \frac{W{v_1}^2}{2g} = \frac{W\cancel{2g}h_1}{\cancel{2g}} = Wh_1$

Total energy = $\cancel{Wh_1}$ + Wh – $\cancel{Wh_1}$ = Wh --- (2)

Case 3: Let the body reaches the ground with final velocity 'v'

Change in K.E = $\frac{Wv^2}{2g} - \frac{W{v_0}^2}{2g} = \frac{Wv^2}{2g}$

But we know that $v^2 - v_0^2 = 2gh$

Therefore $v^2 = 2gh$

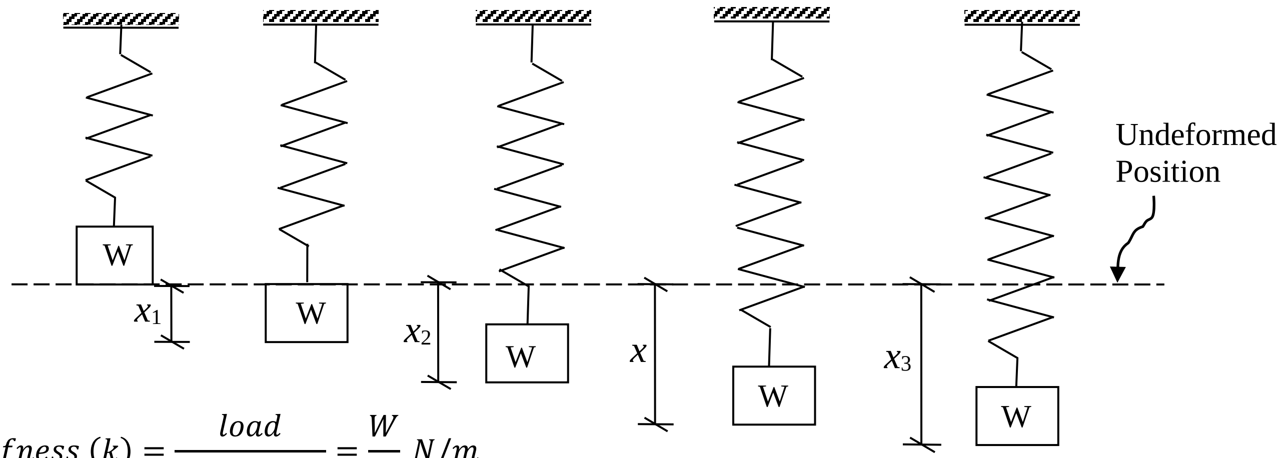
$\therefore \frac{Wv^2}{2g} = \frac{W\cancel{2g}h}{\cancel{2g}} = Wh$

P.E = 0

Therefore Total energy = K.E + P.E = Wh + 0 = Wh--- (3)

From equations (1), (2) and (3) it is clear that "Total energy is constant".

Work done by a spring:



$$stiffness\ (k) = \frac{load}{deflection} = \frac{W}{x}\ N/m$$

W = kx ---- (1)

At any instant, if displacement is dx,

Work done by spring = -W.dx = - k.x.dx

Negative sign indicates force in spring is in opposite direction to that of displacement.

Work done by spring in displacing from x_1 to x_2 is

$$U = \int_{x_1}^{x_2} -kx.dx = -k\left[\frac{x^2}{2}\right]_{x_1}^{x_2}$$

$$U = -\frac{1}{2}k({x_2}^2 - {x_1}^2)$$

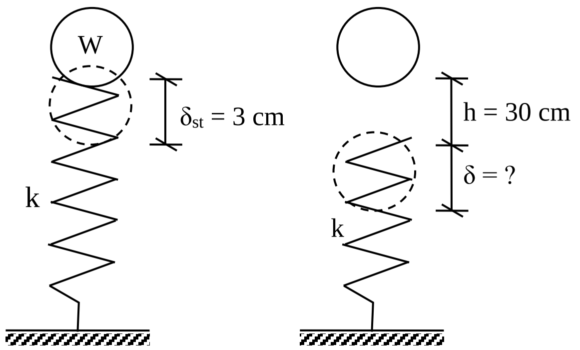
If $x_1 = 0$ (initial position) and displaced by 'x',

Work done by spring U = - $\frac{1}{2}$ kx^2

Ex: When a ball of weight W rests on a spring of constant k, it produced a static deflection of 3 cm. How much will the same ball compress the spring if it is dropped from a height of 30 cm? Neglect mass of the spring.

Sol: Stiffness of the spring $k = \frac{W}{\delta_{st}}$

$k = \frac{W}{3}$ kg / cm



When it is dropped through 30 cm height,

Energy in the spring = $\frac{1}{2}k\delta^2$

Work done by W = W (h + δ) = W (30 + δ)

Change in K.E = work done

$\frac{1}{2}k\delta^2 = W(30 + \delta)$

$\frac{1}{2}\frac{\cancel{W}}{3}\delta^2 = \cancel{W}(30 + \delta)$

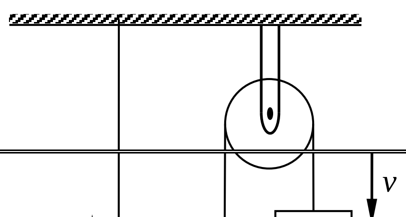
$\delta^2 - 6\delta - 180 = 0$

By solving the above equation we get δ = 16.9 cm

Ex: If a system shown in fig. is released from rest in the configuration shown, find the velocity 'v' of the block as a function of distance 'x' that it falls.

Sol: Consider a system with two connected particles.

Let Q has velocity 'v' then, P moves up by $\frac{v}{2}$



Total K.E = $\frac{1}{2}\frac{Q}{g}v^2 + \frac{P}{2g}\left(\frac{v}{2}\right)^2 \quad = \frac{v^2}{2g}\left[Q + \frac{P}{4}\right] --- (1)$

When the system is released from rest, total change in K.E is equal to the corresponding work done by the above active forces. Let 'x' be the distance travelled by 'Q'

$$\frac{v^2}{2g}\left[Q+\frac{P}{4}\right] = Q.x - P\left(\frac{x}{2}\right)$$

$$\frac{v^2}{2g}\left[Q+\frac{P}{4}\right] = x\left(Q-\frac{P}{2}\right)$$

$$\therefore v = \sqrt{\frac{2gx\left(Q-\frac{P}{2}\right)}{\left(Q+\frac{P}{4}\right)}}$$

Ex: Calculate the velocity 'v' of a 40 kg mass, when it travels down the 20^0 incline for 16 m at B; if it is given an initial velocity of 3 m/sec at A. The coefficient of friction between the mass and the incline is 0.2.

Sol: W = mg = 40 × 9.81 = 392.4 N

N = W cos20^0 = 392.4 cos20^0 = 368.44 N

Frictional force = μN = 0.2 × 368.7 = 73.7 N.

Weight component = 392.4 sin20^0 = 134.21

Net force acting = 134.31 – 73.75 = 60.46 N

Total work done = 60.46 × 16 = 967.4 joules

Change in K.E =

$\frac{m}{2}(v^2 - {v_0}^2) = \frac{40}{2}(v^2 - 3^2) = 20(v^2 - 9)$

Change in K.E = work done

$20(v^2 - 9) = 967.4$

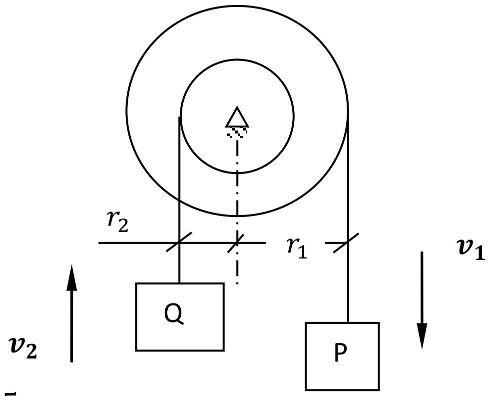
$v^2 = \frac{967.4}{20} + 9 = 57.37$

$\therefore v = \sqrt{57.37} = 7.57\ m/s.$

Ex: If the system in the fig. is released from rest in the configuration shown, find the velocity 'v' of the falling weight P as a function of its displacement x. neglect friction and inertia of the pulleys and assume the following data. P = Q = 44.5 N, r_1 = 150 mm, r_2 = 100 mm and x = 3 m.

Sol: Ratio of displacements of block P & Q

Let us consider P as 1& Q as 2



$\frac{x_1}{x_2} = \frac{\cancel{2\pi} r_1}{\cancel{2\pi} r_2}$

$x_2 = \frac{100}{150}x_1 = \frac{2}{3}x_1$

Work done by the system = P. x_1 - Q. x_2

= 44.5 $(x_1 - \frac{2}{3}x_1)$

= 14.83 x_1 = 14.83 × 3 = 44.5 N-m

Being at rest initially, the initial K.E is zero.

When the block moves downward by 3m, $v_2 = \frac{v_1}{1.5}$ or $v_1 = 1.5v_2$

Final K.E of the system = $\left(\frac{P{v_1}^2}{2g} - 0\right) + \left(\frac{Q{v_2}^2}{2g} - 0\right) = \frac{P{v_1}^2}{2g} + \frac{Q{v_2}^2}{2g}$

= $\frac{44.5}{2\times 9.81}[{v_1}^2 + (\frac{v_1}{1.5})^2$ = 2.268(1.444 ${v_1}^2$)

= 3.276 ${v_1}^2$

But final K.E = work done by the system

3.276 ${v_1}^2$ = 44.5 N-m

$$\therefore v_1 = \sqrt{\frac{44.5}{3.276}} = 3.686\ m/s$$

Ex: A 300N block shown in Fig.06. rests on the smooth plane .it is attached by a flexible inextensible cord over friction and weightless pulley to 450N weight and a support .In what distance will the block on plane attain a speed of 3 m/sec?

Sol: Let v – velocity of 300 N and $\frac{v}{2}$ - velocity of 450 N

From pulley configuration, 450N weight moves half the distance travelled by 300 N block.

There is no friction

Total change in K.E

$$= \left(\frac{W_1 v_1^2}{2g} - \frac{W_1 v_0^2}{2g}\right) + \left(\frac{W_2 v_2^2}{2g} - \frac{W_2 v_0^2}{2g}\right)$$

$$= \frac{300}{2g} \times v^2 + \frac{450}{2g} \times \left(\frac{v}{2}\right)^2$$

$$= \frac{300}{2 \times 9.81} \times 3^2 + \frac{450}{2 \times 9.81} \times \left(\frac{3}{2}\right)^2$$

= 137.6 + 51.6 = 189.2 J

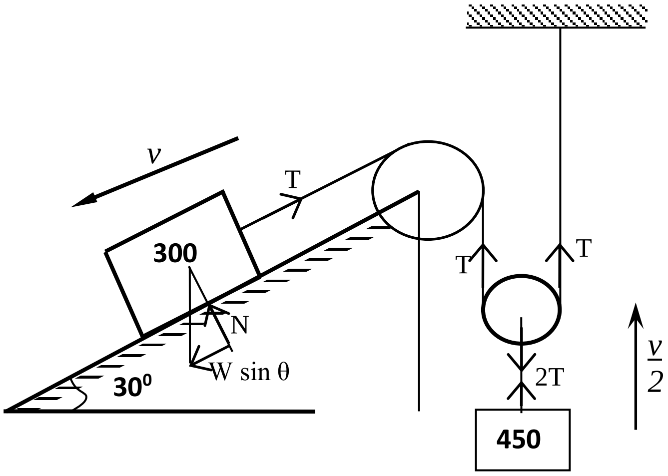
Work done = $W_1 \sin 30^0 . S - 450 \times \frac{s}{2}$

= s (150 – 225)

Equating work done to total change in K.E

s (150 – 225) = 189.2

s = - 2.52 m.



Since the answer is negative, indicates the 450 N block moves down and 300 N block moves up the plane.